12. Before proceeding to the general resolution of the quartic Z, we can utilise the preceding analysis to the construction of some cases of algebraical motion, in the manner explained in the *American Journal of Mathematics*, XX, §33, by taking a parameter

$$v = \frac{2\omega_3}{\mu}, \quad \mu = 2n + 1, \tag{1}$$

and putting

$$\left\{\begin{aligned} \psi - pt &= \psi', \\ &= \frac{1}{2n+1}\cos^{-1}\frac{Hz^{2n+1} + H_1 z^{2n} + \dots}{(-z^2-1)^{n+\frac{1}{2}}} \\ &= \frac{1}{2n+1}\sin^{-1}\frac{Kz^{2n-1} + K_1 z^{2n-2} + \dots}{(-z^2+1)^{n+\frac{1}{2}}}\sqrt{Z}, \end{aligned}\right. \tag{2}$$

and then determining the various constants of the problem.

But the degree of the relation (2) can be halved by changing the variable from $z = \cos \vartheta$ to $t = \tan \frac{1}{2}\vartheta$, a different t from that representing time, henceforth to be distinguished when necessary by an accent, as t'.

This is equivalent to a change to the stereographic projection; and now with

$$z = \frac{1-t^2}{1+t^2}, \tag{3}$$

we can put

$$\left\{\begin{aligned} \psi' &= \frac{1}{n+\frac{1}{2}}\cos^{-1}\frac{ht^{2n-1} + h_1 t^{2n-2} + \dots}{t^{n+\frac{1}{2}}}\sqrt{T_1} \\ &= \frac{1}{n+\frac{1}{2}}\sin^{-1}\frac{ht^{2n-1} - h_1 t^{2n-2} + \dots}{t^{n+\frac{1}{2}}}\sqrt{T_2}, \end{aligned}\right. \tag{4}$$

making

$$\frac{d\psi'}{dt} = -\frac{(L+L')t^4 + 2Pt^2 + L - L'}{t\sqrt{T_1 T_2}}, \tag{5}$$

where

$$\left\{\begin{aligned} T_1 T_2 &= \tfrac{1}{4}(t^2+1)^4 M^2 Z = -(t^2+1)^2\{(L+L')t^2 + L - L'\}^2, \\ &\quad + M^2 t^2\left[-a(t^2-1)^2 - 4\frac{aB}{M}(t^4-1) + (a+D)(t^2+1)^2\right] \\ &= -(L+L')^2 t^8 + 4bt^6 + 6ct^4 + 4dt^2 - (L-L')^2, \end{aligned}\right. \tag{6}$$

suppose; and the troublesome determination of the homogeneity factor M is delayed, and does not hinder us.

To obtain the factors T_1 and T_2, write

(7) $$T_1T_2 = -[(L+L')t^4 + 2\gamma t^2 + L - L']^2 + 4t^2(\beta t^2+\delta)^2,$$

so that

(8) $$T_1 = \quad (L+L')t^4 + 2\beta t^3 + 2\gamma t^2 + 2\delta t + L - L',$$

(9) $$T_2 = -(L+L')t^4 + 2\beta t^3 - 2\gamma t^2 + 2\delta t - L + L',$$

and

(10) $b = \beta^2 - (L+L')\gamma, \quad d = \delta^2 - (L-L')\gamma, \quad 6c = 8\beta\delta - 4\gamma^2 - 2(L^2 - L'^2).$

Denoting the elliptic argument of the motion by u, such that

(11) $$u = \int \frac{dt^2}{\sqrt{(T_1T_2)}},$$

and making use of the Biermann-Weierstrass formula (2), §7, with the argument v_1 and v_2 corresponding now with $t^2 = 0$ and ∞, we find for $v_1 + v_2 = v$,

(12) $$12\wp v = 6c - 6(L^2 - L'^2) = 8\beta\delta - 4\gamma^2 - 8(L^2 - L'^2).$$

The formula

(13) $$\begin{cases} \wp'(v_1+v_2) = -\dfrac{(ay^3+3by^2+3cy+d)x + by^3 + 3cy^2 + 3dy + e}{(x-y)^3}\sqrt{X} \\ \qquad\qquad + \dfrac{(ax^3+3bx^2+3cx+d)y + bx^3 + 3cx^2 + 3dx + e}{(x-y)^3}\sqrt{Y}, \end{cases}$$

with x corresponding to $t^2 = 0$, and y to $t^2 = \infty$, leads to

(14) $$\begin{cases} i\wp' v = -(L-L')b - (L+L')d \\ \quad = -(L-L')\beta^2 - (L+L')\delta^2 + 2(L^2 - L'^2)\gamma. \end{cases}$$

The quadrinvariant

(15) $$\begin{cases} g_2 = ae - 4bd + 3c^2 \\ \quad = (L^2 - L'^2)^2 - 4\{\beta^2 - (L+L')\gamma\}\{\delta^2 - (L-L')\gamma\} \\ \qquad\qquad + \tfrac{1}{12}\{8\beta\delta - 4\gamma^2 - 2(L^2 - L'^2)\}^2, \end{cases}$$

so that

(16) $2\wp'' v = 12\wp^2 v - g_2 = 4(\beta\delta - L^2 + L'^2)^2 + 4\gamma i\wp' v = (\gamma^2 + 3\wp v)^2 + 4\gamma i\wp' v,$

a quartic equation for the determination of γ.

A root of this equation is

(17) $$\gamma = -\tfrac{1}{2}\frac{\wp''\tfrac{1}{2}v}{i\wp'\tfrac{1}{2}v}, \quad \text{the others being} -\tfrac{1}{2}\frac{\wp''(\tfrac{1}{2}v+\omega)}{i\wp'(\tfrac{1}{2}v+\omega)};$$

so that, taking

(18) $$\tfrac{1}{2}v = \frac{2\omega_3}{2n+1}, \quad v = \frac{4\omega_3}{2n+1},$$

and employing Halphen's x and y, we can put (*L. M. S.*, XXV, p. 198)

(19) $$12\wp v = -(y+1)^2 + 8x, \qquad i\wp' v = xy, \qquad \wp'' v = 2x^2 - xy(y+1);$$

(20) $$12\wp\tfrac{1}{2}v = -(y+1)^2 - 4x, \quad i\wp'\tfrac{1}{2}v = x, \quad \wp''\tfrac{1}{2}v = x(y+1),$$

and then

(21) $$\gamma = -\tfrac{1}{2}(y+1),$$

(22) $$\gamma^2 + 3\wp v = 2x,$$

(23) $$\beta\delta = L^2 - L'^2 + x = s + x,$$

putting

(24) $$L^2 - L'^2 = s, \text{ i. e. } s - \sigma, \text{ with } \sigma = 0.$$

(25) $$\begin{cases} (L-L')\beta^2 + (L+L')\delta^2 = -(y+1)(L^2-L'^2) - xy \\ \qquad\qquad = -(y+1)s - xy, \end{cases}$$

(26) $$\{\sqrt{(L-L')}\beta \pm \sqrt{(L+L')}\delta\}^2 = \pm 2s^{\frac{1}{2}}(s+x) - (y+1)s - xy = S_1 \text{ or } S_2,$$

where

(27) $$S_1 S_2 = -4s(s+x)^2 + \{(y+1)s + xy\}^2 = -S;$$

and then

(28) $$\beta = \frac{\sqrt{S_1} + \sqrt{S_2}}{2\sqrt{(L-L')}}, \quad \delta = \frac{\sqrt{S_1} - \sqrt{S_2}}{2\sqrt{(L'+L)}}.$$

With s negative and

(29) $$L'^2 - L^2 = -s = p^2,$$

(30) $$-(L'-L)\beta^2 + (L'+L)\delta^2 = (y+1)p^2 - xy,$$

(31) $$[(L'-L)\beta^2 + (L'+L)\delta^2]^2 = 4p^2(p^2-x)^2 + [(y+1)p^2 - xy]^2 = p'^2,$$

suppose,

(32) $$\begin{matrix}\beta\\ \delta\end{matrix} = \sqrt{\frac{p' \mp [(y+1)p^2 - xy]}{2(L' \mp L)}}.$$

Try, as a special case,

(33) $$s + x = 0, \quad \beta\delta = 0, \quad \beta = 0 \text{ or } \sqrt{\frac{x}{L'-L}}, \quad \delta = \sqrt{\frac{x}{L'+L}} \text{ or } 0,$$

(34) $$v' = \frac{2\omega_3}{\mu}, \quad v_1 = \frac{3\omega_3}{\mu}, \quad v_2 = \frac{\omega_3}{\mu},$$

so that it should be possible theoretically to express α_1, β_1, γ_1, δ_1, by the μth root of an algebraical function.

13. Returning to the original variable z by putting

(1) $$t^2 = \frac{1-z}{1+z},$$

(2) $$\begin{cases} M^2 Z = -\tfrac{1}{4}[(L+L')(1-z)^2 + 2\gamma(1-z^2) + (L-L')(1+z^2)]^2 \\ \qquad\qquad + (1-z^2)[\beta(1-z)+\delta(1+z)]^2 \\ \quad = -[2(L-L'z)-(L-\gamma)(1-z^2)]^2 \\ \qquad\qquad + (1-z^2)[\beta+\delta-(\beta-\delta)z]^2, \end{cases}$$

(3) $$\begin{cases} M^2 Z + 4(L-L'z)^2 = (1-z^2)\{4(L-\gamma)(L-L'z)-(L-\gamma)^2(1-z^2) \\ \qquad\qquad + (\beta+\delta)^2 - 2(\beta^2-\delta^2)z + (\beta-\delta)^2 z^2\}, \end{cases}$$

so that

(4) $$M^2 a = (\beta-\delta)^2 + (L-\gamma)^2,$$

implying $a = -1$ and an oblate figure for real values,

(5) $$2aBM = -(\beta^2-\delta^2) - 2L'(L-\gamma),$$

(6) $$M^2(a+D) = (\beta+\delta)^2 - (L-\gamma)^2 + 4L(L-\gamma).$$

Now with $a = -1$,

(7) $$M^2 = L^2 - 2s + \tfrac{1}{4}(y+1)^2 - 2x + \frac{L'\surd(-S) - Lxy}{s},$$

(8) $$2BM = 2LL' + \frac{L\surd(-S) - L'xy}{s},$$

(9) $$M^2(1-D) = -3L^2 - 2s + \tfrac{1}{4}(y+1)^2 - 2x - \frac{L'\surd(-S) - Lxy}{s},$$

(10) $$M^2 D = 2\frac{L'\surd(-S) - Lxy}{s} + 4L^2,$$

(11) $$M^2 E = 2\frac{L'\surd(-S) - xyL}{s} + 4L'^2,$$

(12) $$\begin{cases} M^2\rho^2 = M^2\left(z^2 - 4\dfrac{B}{M}z - 1 + E\right) \\ \qquad = [(\beta-\delta)z + \beta + \delta]^2 + [(L-\gamma)z + 2L']^2 - (L+\gamma)^2, \end{cases}$$

(13) $$\frac{\Delta}{M^2} = L^2 - 2s + \tfrac{1}{4}(y+1)^2 - 2x - \frac{Lxy}{s} - \frac{L'\surd(-S)}{s},$$

(14) $$\begin{cases} \Delta = \left[L^2 - 2s + \tfrac{1}{4}(y+1)^2 - 2x - \dfrac{Lxy}{s}\right]^2 + \dfrac{L'^2 S}{s^2} \\ \quad = \dfrac{-AL'^2 + B}{s}, \end{cases}$$

(15) $$A = (L^2 + 3\wp v)^2 + 4Li\wp' v - 2\wp v'',$$

(16) $$B = (L^3 + 3L\wp v + i\wp' v)^2.$$

Putting

(17) $L + \frac{1}{2}(y+1) = \lambda,$

(18) $$A = \lambda\left(\lambda - \tfrac{1}{2}\sqrt{\frac{\sigma_2\sigma_3}{-\sigma_1}}\right)\left(\lambda - \tfrac{1}{2}\sqrt{\frac{-\sigma_1\sigma_3}{\sigma_2}}\right)\left(\lambda - \tfrac{1}{2}\sqrt{\frac{-\sigma_1\sigma_2}{\sigma_3}}\right),$$

(19) $$\sigma_a = s - s_a.$$

In Kirchhoff's special case, considered in the *American Journal of Mathematics*, XX, $B = 0$, and therefore

(20) $$\left(2LL' - \frac{L'xy}{s}\right)^2 + \frac{L^2S}{s^2} = 0,$$

(21) $$L'^2 = \frac{4L^2(L^2+x)^2 - [(y+1)L^2 + xy]^2}{4L[L^3 - \{\frac{1}{4}(y+1)^2 - 2x\}L + xy]},$$

(22) $$LM^2 = -L^3 + \{\tfrac{1}{4}(y+1)^2 - 2x\}L - xy,$$

(23) $$4L^2L'^2M^2 = -4L^2(L^2+x)^2 + [(y+1)L^2 + xy]^2,$$

(24) $$LL'M = aN_1N_2N_3,$$

as in the *American Journal of Mathematics*, XX (59).

14. We can illustrate the preceding theory immediately for the simplest values of μ, utilising the analysis in the *Phil. Trans.*, 1904.

(1) $\mu = 3, \quad v = \frac{4}{3}\omega_3, \quad x = 0,$

(2) $\beta\delta = L^2 - L'^2, \quad \beta = (L+L')m, \quad \delta = \dfrac{L-L'}{m},$

(3) $T_1, T_2 = \pm(L+L')t^4 + 2(L+L')mt^3 \pm 2\gamma t^2 + 2\dfrac{L-L'}{m}t \pm (L-L'),$

and then

(4) $$\psi' = \tfrac{2}{3}\cos^{-1}\frac{t-m}{Nt^{\frac{3}{2}}}\sqrt{T_1} = \tfrac{2}{3}\sin^{-1}\frac{t+m}{Nt^{\frac{3}{2}}}\sqrt{T_2},$$

where

(5) $$N^2 = 4m\left[(L+L')m^2 - 2\gamma + \frac{L-L'}{m^2}\right],$$

leads on differentiation to

(6) $$\frac{d\psi}{dt} = -\frac{(L+L')t^4 + \frac{2}{3}\gamma t^2 + L - L'}{t\sqrt{(T_1T_2)}},$$

and so provides an algebraical case.

For higher values of $\mu = 2n + 1$, the work is rendered more symmetrical by the use of a new variable q, defined by

$$t = q \sqrt[4]{\frac{L - L'}{L + L'}}, \text{ or } q \sqrt[4]{\frac{L' - L}{L' + L}}, \tag{7}$$

according as s is positive $= p^2$, or negative $= -p^2$.

Now with s positive and

$$\frac{T_1}{L - L'} = Q_1, \quad \frac{T_2}{L - L'} = Q_2, \tag{8}$$

$$\begin{matrix} Q_1 \\ Q_2 \end{matrix} = \pm q^4 + \frac{\sqrt{S_1} + \sqrt{S_2}}{p^{\frac{3}{2}}} q^3 \mp \frac{y+1}{p} q^2 + \frac{\sqrt{S_1} - \sqrt{S_2}}{p^{\frac{3}{2}}} q \pm 1, \tag{9}$$

$$\begin{matrix} S_1 \\ S_2 \end{matrix} = \pm 2p(p^2 + x) - (y + 1)p^2 - xy, \tag{10}$$

we have to satisfy the relation

$$\frac{d\psi'}{dq} = -\frac{q^4 + 2q^2 \dfrac{P}{p} + 1}{q\sqrt{Q_1 Q_2}}, \tag{11}$$

by an expression of the form

$$\left\{\begin{aligned} \psi' &= \frac{1}{n + \frac{1}{2}} \cos^{-1} \frac{Aq^{2n-1} + Bq^{2n-2} + \dots + Kq + L}{Nq^{n+\frac{1}{2}}} \sqrt{Q_1}, \\ &= \frac{1}{n + \frac{1}{2}} \sin^{-1} \frac{Aq^{2n-1} - Bq^{2n-2} + \dots + Kq - L}{Nq^{n+\frac{1}{2}}} \sqrt{Q_2}, \end{aligned}\right. \tag{12}$$

so that, differentiating, the coefficients $A, B, \dots, K, L$ (the two L's must be kept distinct), are obtained from the identity

$$\left\{\begin{aligned} &[(2n-3)Aq^{2n-1} + (2n-5)Bq^{2n-2} + \dots - (2n-1)Kq - (2n+1)L] \\ &\qquad \left(q^4 + \frac{\sqrt{S_1} + \sqrt{S_2}}{p^{\frac{3}{2}}} q^3 - \frac{y+1}{p} q^2 + \frac{\sqrt{S_1} - \sqrt{S_2}}{p^{\frac{3}{2}}} q + 1\right) \\ &+ (Aq^{2n-1} + Bq^{2n-2} + \dots + Kq + L) \\ &\qquad \left(4q^4 + 3\frac{\sqrt{S_1} + \sqrt{S_2}}{p^{\frac{3}{2}}} q^3 - 2\frac{y+1}{p} q^2 + \frac{\sqrt{S_1} - \sqrt{S_2}}{p^{\frac{3}{2}}} q + 0\right) \\ &= (Aq^{2n-1} - Bq^{2n-2} + \dots + Kq - L) \\ &\qquad \left[(2n+1)q^4 + 2q^2 \frac{(2n+1)P}{p} + 2n + 1\right]. \end{aligned}\right. \tag{13}$$

Hence

(14) $$\frac{B}{A} = -\frac{\sqrt{S_1}+\sqrt{S_2}}{2p^{\frac{3}{2}}}, \quad \frac{K}{L} = -\frac{\sqrt{S_1}-\sqrt{S_2}}{2p^{\frac{3}{2}}},$$

(15) $$\begin{cases} \dfrac{C}{A} = -(n-1)\dfrac{(\sqrt{S_1}+\sqrt{S_2})^2}{4p^3} - \dfrac{(2n-1)(y+1)+(4n+2)P}{4p} \\ \dfrac{H}{L} = -(n-1)\dfrac{(\sqrt{S_1}-\sqrt{S_2})^2}{4p^3} - \dfrac{(2n-1)(y+1)+(4n+2)P}{4p}, \end{cases}$$

and so on.

Induction leads us to infer that

(16) $$\frac{L}{A} = e^{(n+\frac{1}{2})I(v)} = \frac{E_1\sqrt{S_1}+E_2\sqrt{S_2}}{2p^{n+\frac{1}{2}}},$$

derived from the associated pseudo-elliptic integral

(17) $$\begin{cases} I(v) = \displaystyle\int \frac{Ps - \frac{1}{2}xy}{s}\frac{ds}{\sqrt{(-S)}} \\ = \dfrac{1}{n+\frac{1}{2}}\,\mathrm{ch}^{-1}\dfrac{E_1\sqrt{S_1}}{2p^{n+\frac{1}{2}}} = \dfrac{1}{n+\frac{1}{2}}\,\mathrm{sh}^{-1}\dfrac{E_2\sqrt{S_2}}{2p^{n+\frac{1}{2}}}, \end{cases}$$

so that we may take A and L as reciprocal, and $AL = -1$,

(18) $$\frac{A}{L} = \pm\sqrt{\frac{E_1\sqrt{S_1} \mp E_2\sqrt{S_2}}{2p^{n+\frac{1}{2}}}},$$

and thence $\dfrac{B}{K}, \dfrac{C}{H}, \ldots\ldots$

In the second case of s negative we have to change p^2 into $-p^2$, and q^4 into $-q^4$, and take

(19) $$\frac{T_1}{L'-L} = Q_1, \quad \frac{T_2}{L'-L} = Q_2,$$

(20) $$\begin{matrix} Q_1 \\ Q_2 \end{matrix} = \pm q^4 + q^3\sqrt{\frac{p'+R}{2p^{\frac{3}{2}}}} \mp q^2\frac{y+1}{p} + q\sqrt{\frac{p'-R}{2p^{\frac{3}{2}}}} \mp 1,$$

where

(21) $$p'^2 = 4p^2(p^2-x)^2 + [(y+1)p^2 - xy]^2, \quad R = -(y+1)p^2 + xy.$$

Tested in this way for $\mu = 3$,

(22) $$\begin{cases} \psi' = \frac{1}{3}\cos^{-1}\dfrac{(Aq+L)\sqrt{Q_1}}{Nq^{\frac{3}{2}}} \\ = \frac{1}{3}\sin^{-1}\dfrac{(Aq-L)\sqrt{Q_2}}{Nq^{\frac{3}{2}}}, \end{cases}$$

with

(23) $$\frac{Q_1}{Q_2} = \pm q^4 + \frac{\sqrt{S_1} + \sqrt{S_2}}{p^{\frac{3}{2}}} q^3 \pm \frac{q^2}{p} + \frac{\sqrt{S_1} - \sqrt{S_2}}{p^{\frac{3}{2}}} q \pm 1,$$

(24) $$\frac{S_1}{S_2} = \pm 2p^3 - p^2 + c,$$

(25) $$\frac{A}{L} = \pm \sqrt{\frac{\sqrt{S_1} \mp \sqrt{S_2}}{2p^{\frac{3}{2}}}},$$

(26) $$N^2 = \frac{4c}{p^3},$$

leading to

(27) $$\frac{d\psi'}{dq} = -\frac{q^4 + \frac{q^2}{3p} + 1}{q\sqrt{Q_1 Q_2}}.$$

Tested by $\mu = 5$, we find for

(28) $$v = \tfrac{4}{5}\omega_3, \quad x = y, \quad P = \frac{1 - 3x}{10},$$

(29) $$\frac{S_1}{S_2} = \pm 2p^3 - (1 + y)p^2 \pm 2yp - y^2,$$

(30) $$\frac{A}{L} = \pm \sqrt{\frac{(p + y)\sqrt{S_1} \mp (-p + y)\sqrt{S_2}}{2p^{\frac{5}{2}}}}, \quad AL = -1,$$

(31) $$\frac{B}{K} = \mp \sqrt{\frac{-(p^2 + p + y)\sqrt{S_1} \pm (p^2 - p + y)\sqrt{S_2}}{2p^{\frac{7}{2}}}},$$

(32) $$\begin{cases} \psi' = \tfrac{2}{5} \cos^{-1} \dfrac{Aq^3 + Bq^2 + Kq + L}{Nq^{\frac{5}{2}}} \sqrt{Q_1}, \\ \quad = \tfrac{2}{5} \sin^{-1} \dfrac{Aq^3 - Bq^2 + Kq - L}{Nq^{\frac{5}{2}}} \sqrt{Q_2}, \end{cases}$$

(33) $$N^2 = \frac{4y^3}{p^5}.$$

This can also be written

(34) $$\begin{cases} \psi' = \tfrac{2}{5} \cos^{-1} \dfrac{q^3 + B'q^2 + K'q + L'}{N'q^{\frac{5}{2}}} \sqrt{Q_1} \\ \quad = \tfrac{2}{5} \sin^{-1} \dfrac{-q^3 + B'q^2 - K'q + L'}{N'q^{\frac{5}{2}}} \sqrt{Q_2}, \end{cases}$$

(35) $$B' = \frac{B}{A} = -\frac{\sqrt{S_1} + \sqrt{S_2}}{2p^{\frac{3}{2}}},$$

(36) $$K' = \frac{K}{A} = \frac{(-1 + y)p^2 + y^2 - \sqrt{S_1}\sqrt{S_2}}{2p^3},$$

(37) $$L' = \frac{L}{A} = -\frac{(p + y)\sqrt{S_1} + (-p + y)\sqrt{S_2}}{2p^{\frac{5}{2}}}.$$

Tested by $\mu = 7$ (*Phil. Trans.*, §9),

(38) $v = \frac{4}{7}\omega_3, \quad x = z(1-z)^2, \quad y = z(1-z), \quad P = \dfrac{3 - 9z + 5z^2}{14},$

$$(39)\begin{cases} \dfrac{A}{L} = \pm e^{\mp\frac{2}{7}I(v)} \\ \quad = \pm\sqrt{\dfrac{[p^2+(1-z)^2p+z(1-z)^3]\sqrt{S_1} \mp [p^2-(1-z)^2p+z(1-z)^3]\sqrt{S_2}}{2p^{\frac{5}{2}}}}, \end{cases}$$

and thence

$$(40)\quad \frac{B}{K} = -\sqrt{\frac{[p^3-(1-z)^2p^2-(1-z)^3p-z(1-z)^4]\sqrt{S_1} \mp [-p^3-(1-z)^2p^2+(1-z)^3p-z(1-z)^4]\sqrt{S_2}}{2p^{\frac{9}{2}}}}$$

$$(41)\quad \frac{C}{H} = -\sqrt{\frac{[-4p^4+0+(5z^2-4z^3)p^2+z^2(1-z)^2p+z^3(1-z)^3]\sqrt{S_1} \mp [-4p^4+0+(5z^2-4z^3)p^2-z^2(1-z)^2p+z^3(1-z)^3]\sqrt{S_2}}{2p^{\frac{11}{2}}}}$$

in

$$(42)\quad \begin{cases} \psi' = \frac{2}{7}\cos^{-1}\dfrac{Aq^5+Bq^4+Cq^3+Hq^2+Kq+L}{Nq^{\frac{7}{2}}}\sqrt{Q_1} \\ \quad = \frac{2}{7}\sin^{-1}\dfrac{Aq^5-Bq^4+Cq^3-Hq^2+Kq-L}{Nq^{\frac{7}{2}}}\sqrt{Q_2}, \end{cases}$$

$$(43)\quad N^2 = -\frac{4z^5(1-z)^6}{p^7}$$

so that Q_1 and Q_2 must change sign when z is positive.

Similarly for $\mu = 9$, *Phil. Trans.*, §10,

$$(44)\begin{cases} v = \frac{4}{9}\omega_3, \quad x = c^2(1-c)(1-c+c^2), \quad y = c^2(1-c), \\ \qquad 18P = 1 + 0 - 3c^2 + 7c^3 \end{cases}$$

$$(45)\quad \frac{A}{L} = \mp\sqrt{\frac{(p^3+h_1p^2+h_2p+h_3)\sqrt{S_1} \mp (-p^3+h_1p^2-h_2p+h_3)\sqrt{S_2}}{2p^{\frac{9}{2}}}},$$

(46) $h_1 = c^2(1-2c), \quad h_2 = -c^4(1-c+c^2), \quad h_3 = -c^6(1-c)(1-c+c^2);$

and thence the remaining coefficients

$$(47)\quad \frac{B}{K} = \mp\sqrt{\frac{(-p^4+k_1p^3+k_2p^2+k_3p+k_4)\sqrt{S_1} \mp (-p^4-k_1p^3+k_2p^2-k_3p+k_4)\sqrt{S_2}}{2p^{\frac{11}{2}}}},$$

$$(48)\quad \begin{cases} k_1 = -(1+c^3), \quad k_2 = -c^2(1-c+c^2)(1-c-c^2), \\ \qquad k_3 = c^4(1-c+c^2)^2, \quad k_4 = c^6(1-c)(1-c+c^2)^2; \end{cases}$$

and so on, in

$$(49)\quad \begin{cases} \psi' = \frac{2}{9}\cos^{-1}(Aq^7 + Bq^6 + Cq^5 + Dq^4 + Gq^3 + Hq^2 + Kq + L)\sqrt{Q_1} \div Nq^{\frac{9}{2}} \\ \quad = \frac{2}{9}\sin^{-1}(Aq^7 - Bq^6 + Cq^5 + Dq^4 + Gq^3 - Hq^2 + Kq - L)\sqrt{Q_2} \div Nq^{\frac{9}{2}}, \end{cases}$$

$$(50)\qquad N = \frac{2c^2(1-c)^2(1-c+c^2)}{p^{\frac{3}{2}}}.$$

Material for the construction of the case of $\mu = 11, 13, 15, 17, \ldots,$ will be found in *Phil. Trans.*, §§11, 12, 13, 15, ...

15. With a parameter v a fraction of the imaginary period ω_3 we must take $a = -1$, as stated above in (4), §13; this is also evident from the graph of the quartic Z, which must now have all its roots included between $z = -1$ and $+1$.

At the same time as ψ' is given in (2), §12, $\varpi' = \varpi - pt$ is given by a relation of similar form,

$$(1)\quad \begin{cases} \varpi' = \dfrac{1}{2n+1}\cos^{-1}\dfrac{Hz^{2n+1} + H_1z^{2n} + \ldots}{\rho^{2n+1}} \\ \quad = \dfrac{1}{2n+1}\sin^{-1}\dfrac{Kz^{2n-1} + K_1z^{2n-2} + \ldots}{\rho^{2n+1}}\sqrt{Z}, \end{cases}$$

where, as in (7), §5,

$$(2)\qquad \rho^2 = \left(z - 2\frac{B}{M}\right)^2 - k, \quad k = 1 + aE + 4\frac{B^2}{M^2}.$$

If k is positive, denote it by λ^2, and put

$$(3)\qquad z - 2\frac{B}{M} = \lambda\frac{v^2+1}{v^2-1}, \quad dz = -\frac{4\lambda v dv}{(v^2-1)^2},$$

$$(4)\qquad \rho = \lambda\frac{2v}{v^2-1},$$

where this new v must not be confused with the parameter v.

The degree can now be halved, as before in §12 for ψ', so that

$$(5)\quad \begin{cases} \varpi' = \dfrac{1}{n+\frac{1}{2}}\cos^{-1}\dfrac{kv^{2n-1} + k_1v^{2n-2} + \ldots}{Nv^{n+\frac{1}{2}}}\sqrt{V_1} \\ \quad = \dfrac{1}{n+\frac{1}{2}}\sin^{-1}\dfrac{kv^{2n-1} - k_1v^{2n-2} + \ldots}{Nv^{n+\frac{1}{2}}}\sqrt{V_2}, \end{cases}$$

which is required to satisfy the differential relation, from (12), §4,

$$(6)\quad \left\{\begin{aligned} d\varpi' &= \left[2\frac{\left(z-2\frac{B}{M}\right)\left(\frac{Lz-L'}{M}\right)}{\left(z-2\frac{B}{M}\right)^2-k} - \frac{L-P}{M}\right]\frac{dz}{\sqrt{Z}} \\ &= \frac{\frac{L+P}{M}\left(z-2\frac{B}{M}\right)^2 + 2\frac{L''\lambda}{M}\left(z-2\frac{B}{M}\right) + \frac{L-P}{M}\lambda^2}{\left(z-2\frac{B}{M}\right)^2-k}\frac{dz}{\sqrt{Z}}, \end{aligned}\right.$$

where

$$(7)\qquad 2\frac{BL}{M} - L' = L''\lambda;$$

and thence

$$(8)\qquad \frac{d\varpi'}{dv} = -\frac{(L+L'')v^4 + 2Pv^2 + L - L''}{v\sqrt{(V_1V_2)}},$$

where

$$(9)\qquad M^2Z = \frac{4\lambda^2 V_1 V_2}{(v^2-1)^4},$$

a relation similar to (6), §12.

Writing the quartic Z in the form (3), §3, we saw in (14), §7, that the elliptic arguments v_3, v_4 corresponding to

$$(10)\qquad \rho = 0,\quad z = 2\frac{B}{M} \pm \lambda,$$

were such that

$$(11)\qquad v_3 + v_4 = v = v_1 + v_2.$$

But written in the form

$$(12)\qquad Z = a(z^2 - F)\left[\left(z - 2\frac{B}{M}\right)^2 - \lambda^2\right] - 4(Hz - K)^2,$$

the values $z = \pm\sqrt{F}$ correspond to arguments v_5 and v_6, such that

$$(13)\qquad v_5 - v_6 = v_3 - v_4.$$

Equating the coefficients in the two forms of Z,

$$(14)\qquad 4H^2 = -a(F-1) + 4\frac{L^2}{M^2},$$

$$(15)\qquad 4HK = -2a(F-1)\frac{B}{M} + 4\frac{LL'}{M^2},$$

$$(16)\qquad 4K^2 = a(F-1)\left(4\frac{B^2}{M^2} - k\right) + 4\frac{L'^2}{M^2};$$

and eliminating H and K, we obtain the equation for F, such that, rejecting $F-1=0$,

$$(17)\qquad F-1=4a\frac{L^2-L''^2}{M^2},$$

$$(18)\qquad Hz-K=\frac{L''}{M}\left(z-2\frac{B}{M}\right)+\frac{L\lambda}{M}.$$

Then

$$(19)\qquad Z+4\left[\frac{L''}{M}\left(z-2\frac{B}{M}\right)+\frac{L\lambda}{M}\right]^2=a(z^2-F)\left[\left(z-2\frac{B}{M}\right)^2-\lambda^2\right],$$

and introducing an arbitrary α,

$$(20)\quad\left\{\begin{aligned}&\lambda^2Z+\left\{\alpha\left[\left(z-2\frac{B}{M}\right)^2-\lambda^2\right]-2\frac{L''\lambda}{M}\left(z-2\frac{B}{M}\right)-\frac{2L\lambda^2}{M}\right\}^2\\&\quad=\left[\left(z-2\frac{B}{M}\right)^2-\lambda^2\right]\left\{\alpha^2\left[\left(z-2\frac{B}{M}\right)^2-\lambda^2\right]-4\lambda\alpha\left[\frac{L''}{M}\left(z-2\frac{B}{M}\right)+\frac{L\lambda}{M}\right]\right.\\&\quad\left.+\lambda^2a\left[\left(z-2\frac{B}{M}\right)^2+4\frac{B}{M}\left(z-2\frac{B}{M}\right)+4\frac{B^2}{M^2}-F\right]\right\}.\end{aligned}\right.$$

Taking the second quadratic factor

$$(21)\quad\left\{\begin{aligned}&(\alpha^2+a\lambda^2)\left(z-2\frac{B}{M}\right)^2-4\lambda\left(\frac{L''}{M}\alpha-\frac{aB\lambda}{M}\right)\left(z-2\frac{B}{M}\right)\\&\qquad-\left(\alpha^2+4\frac{L}{M}\alpha-4\frac{aB}{M^2}+aF\right)^2,\end{aligned}\right.$$

make it a square by the condition

$$(22)\qquad(\alpha^2+a\lambda^2)\left(\alpha^2+4\frac{L}{M}\alpha-4\frac{aB^2}{M^2}+aF\right)+4\left(\frac{L''}{M}\alpha-\frac{aB\lambda}{M}\right)^2=0.$$

This quartic equation on putting

$$(23)\qquad \alpha=\frac{\gamma-L}{M},$$

will be found to reduce to

$$(24)\qquad(\gamma^2+3\wp v)^2+4\gamma i\wp' v-2\wp'' v=0,$$

as before in (16), §12.

Then we can write

$$(25)\quad\left\{\begin{aligned}V_1V_2=\tfrac{1}{4}\frac{M^2}{\lambda^2}Z(v^2-1)^4=-[(L+L'')v^4-(1+y)v^2+L-L'']^2\\+4v^2(\beta'v^2+\delta')^2,\end{aligned}\right.$$

of the same form as $T_1 T_2$ in (7), §12, and the resolution is effected in the same way, so that the preceding algebraical results for ψ' can be utilised with slight modification for ϖ'.

When k is negative we must take

$$(26)\qquad z - 2\frac{B}{M} = \lambda\frac{-w^2+1}{2w},\quad \lambda^2 = -k,$$

$$(27)\qquad \rho = \lambda\frac{w^2+1}{2w},$$

to effect the halving of the degree of the algebraical results in the form

$$(28)\qquad \begin{cases} \varpi' = \dfrac{1}{n+\frac{1}{2}}\cos^{-1}\dfrac{Hw^{2n-1}+\dots\dots}{(w^2+1)^{n+\frac{1}{2}}}\surd W_1, \\[2ex] \quad = \dfrac{1}{n+\frac{1}{2}}\sin^{-1}\dfrac{Kw^{2n-1}+\dots\dots}{(w^2+1)^{n+\frac{1}{2}}}\surd W_2, \end{cases}$$

satisfying the differential relation

$$(29)\qquad \frac{d\varpi'}{dw} = -2\frac{(L+P)(w^2+1)^2 - 4L''(w^3-w) - 8Lw^2}{(w^2+1)\surd(W_1W_2)}.$$

We can however infer the result from the form when k is positive by putting

$$(30)\qquad v = \frac{w+1}{-w+1},\quad k = \lambda^2,$$

$$(31)\qquad z - 2\frac{B}{M} = \lambda\frac{v^2+1}{v^2-1} = \lambda\frac{w^2+1}{2w},$$

$$(32)\qquad \rho = \lambda\frac{2v}{v^2-1} = \lambda\frac{-w^2+1}{2w},$$

when

$$(33)\qquad \frac{d\varpi'}{dw} = -2\frac{(L+P)(w^2-1)^2 + 4L''(w^3+w) + 8Lw^2}{(-w^2+1)\surd(W_1W_2)},$$

and then a change of λ into λi, L'' into $L''i$, and w into wi, with appropriate change in the coefficients will enable us to pass from positive to negative k.

Now with positive k, and

$$(34)\qquad V_1 = \alpha v^4 + 2\beta v^3 + 2\gamma v^2 + 2\delta v + \varepsilon,$$

$$(35)\qquad V_2 = -\alpha v^4 + 2\beta v^3 - 2\gamma v^2 + 2\delta v - \varepsilon,$$

and with

$$(36)\qquad \begin{cases} W_1 = (w-1)^4 V_1 \\ \quad\; = aw^4 + 4bw^3 + 6cw^2 + 4dw + e, \end{cases}$$

we find

$$(37)\qquad W_2 = -ew^4 - 4dw^3 - 6cw^2 - 4dw - e.$$

16. In the algebraical case

$$(1)\qquad \begin{cases} \varpi' = \dfrac{1}{n+\frac{1}{2}} \cos^{-1} \dfrac{H_0 w^{2n-1} + H_1 w^{2n-2} \quad + \ldots\ldots + H_{2n-1}}{(-w^2+1)^{n+\frac{1}{4}}} \sqrt{W_1} \\ \quad = \dfrac{1}{n+\frac{1}{2}} \sin^{-1} \dfrac{H_{2n-1} w^{2n-1} + H_{2n-2} w^{2n-2} + \ldots\ldots + H_0}{(-w^2+1)^{n+\frac{1}{4}}} \sqrt{W_2}. \end{cases}$$

Thus for instance, for $\mu = 3$,

$$(2)\qquad \varpi' = \tfrac{2}{3} \cos^{-1} \frac{v-n}{Nv^{\frac{3}{2}}} \sqrt{V_1} = \tfrac{2}{3} \sin^{-1} \frac{v+n}{Nv^{\frac{3}{2}}} \sqrt{V_2},$$

$$(3)\quad V_1, V_2 = \pm(L+L'')v^4 + 2(L+L'')nv^3 \pm 2\gamma v^2 + 2(L-L'')\frac{v}{n} \pm (L-L'')$$

leading on differentiation to

$$(4)\qquad \frac{d\varpi'}{dv} = -\frac{(L+L'')v^4 + \frac{2}{3}\gamma v^2 + L - L''}{v\sqrt{(V_1 V_2)}};$$

and replacing v by the new variable w from (30), §15,

$$(5)\qquad \varpi' = \tfrac{2}{3} \cos^{-1} \frac{Aw+B}{N(-w^2+1)^{\frac{3}{2}}} \sqrt{W_1} = \tfrac{2}{3} \sin^{-1} \frac{Bw+A}{N(-w^2+1)^{\frac{3}{2}}} \sqrt{W_2},$$

where

$$(6)\qquad A = 1+n, \quad B = 1-n,$$

$$(7)\qquad W_1 = aw^4 + 4bw^3 + 6cw^2 + 4dw + e,$$

$$(8)\qquad W_2 = -ew^4 - 4dw^3 - 6cw^2 - 4bw - a,$$

$$(9)\qquad a = 2(L+\gamma) - 2(L+L'')n - 2(L-L'')\frac{1}{n},$$

$$(10)\qquad e = 2(L+\gamma) + 2(L+L'')n + 2(L-L'')\frac{1}{n},$$

$$(11)\qquad b = 2L'' - (L+L'')n + (L-L'')\frac{1}{n},$$

$$(12)\qquad d = 2L'' + (L+L'')n - (L-L'')\frac{1}{n},$$

$$(13)\qquad c = 2L - \tfrac{2}{3}\gamma.$$

Putting $n = e^a$, we can write

(14) $\varpi' = \frac{2}{3}\cos^{-1}\dfrac{w\operatorname{ch}\frac{1}{2}a - \operatorname{sh}\frac{1}{2}a}{N(-w^2+1)^{\frac{3}{2}}}\sqrt{W_1} = \frac{2}{3}\sin^{-1}\dfrac{w\operatorname{sh}\frac{1}{2}a - \operatorname{ch}\frac{1}{2}a}{N(-w^2+1)^{\frac{3}{2}}}\sqrt{W_2}$,

(15) $a = L + \gamma - 2L\operatorname{ch}a - 2L''\operatorname{sh}a$,

(16) $e = L + \gamma + 2L\operatorname{ch}a + 2L''\operatorname{sh}a$,

(17) $b = L'' - L\operatorname{sh}a - L''\operatorname{ch}a$,

(18) $d = L'' + L\operatorname{sh}a + L''\operatorname{ch}a$,

(19) $c = L - \frac{1}{3}\gamma$,

leading on differentiation to (33), §15.

Now to pass to negative k, replace w, λ, L'', a, by wi, λi, $L''i$, ai; so we replace $\operatorname{ch}a$ by $\cos \operatorname{sh}a$, a by $i\sin a$, and put

(20) $W_1 = aw^4 + 4bw^3 + 6cw^2 + 4dw + e$

(21) $W_2 = ew^4 - 4dw^3 + 6cw^2 - 4bw + a$,

with

(22) $a = L + \gamma - 2L\cos a - 2L''\sin a$,

(23) $e = L + \gamma + 2L\cos a + 2L''\sin a$,

(24) $b = -L'' - L\sin a + L''\cos a$,

(25) $d = + L'' - L\sin a + L''\cos a$,

(26) $c = -L + \frac{1}{3}\gamma$;

and now

(27) $\varpi' = \frac{2}{3}\cos^{-1}\dfrac{w\cos\frac{1}{2}a - \sin\frac{1}{2}a}{N(w^2+1)^{\frac{3}{2}}}\sqrt{W_1} = \frac{2}{3}\sin^{-1}\dfrac{w\sin\frac{1}{2}a + \cos\frac{1}{2}a}{N(w^2+1)^{\frac{3}{2}}}\sqrt{W_2}$,

satisfying the relation

(28) $(w\cos\frac{1}{2}a - \sin\frac{1}{2}a)^2 W_1 + (w\sin\frac{1}{2}a + \cos\frac{1}{2}a)^2 W_2 = N^2(w^2+1)^3$,

(29) $N^2 = \gamma - L\cos 2a - L''\sin 2a$;

(30) $\dfrac{d\varpi'}{dw} = -2\dfrac{(L+\frac{1}{3}\gamma)(w^2+1)^2 - 4L''w(w^2-1) - 8Lw^2}{(w^2+1)\sqrt{(W_1W_2)}}$.

We can now infer that the general result in the equation (1), with $k = -\lambda^2$, assumes the form

(31)
$$\begin{cases} \varpi' = \dfrac{1}{n+\frac{1}{2}}\cos^{-1}\dfrac{H_0w^{2n-1} + H_1w^{2n-2} + \dots + H_{2n-1}}{(w^2+1)^{n+\frac{1}{2}}}\sqrt{W_1} \\[2ex] \quad = \dfrac{1}{n+\frac{1}{2}}\sin^{-1}\dfrac{H_{2n-1}w^{2n-1} - H_{2n-2}w^{2n-2} + \dots - H_0}{(w^2+1)^{n+\frac{1}{2}}}\sqrt{W_2}, \end{cases}$$

with W_1 and W_2 related as in (20) and (21).

The details of the calculation for $\mu = 2n + 1 = 5, 7, 9, \dots$, can be left as an exercise.

17. With μ even, $= 2n$, it is not possible to have

$$(1)\qquad \begin{cases} \psi' = \dfrac{1}{n} \cos^{-1} \dfrac{ht^{2n-2} + h_1 t^{2n-3} + \dots}{Nt^n} \sqrt{T_1} \\ \quad = \dfrac{1}{n} \sin^{-1} \dfrac{ht^{2n-2} - h_1 t^{2n-3} + \dots}{Nt^n} \sqrt{T_2}, \end{cases}$$

because, squaring and adding,

$$(2)\qquad (ht^{2n-2} + h_1 t^{2n-3} + \dots)^2 T_1 + (ht^{2n-2} - h_1 t^{2n-3} + \dots)^2 T_2,$$

consists of odd powers of t, and cannot be made equal to $N^2 t^{2n}$.

But in the case of μ even, T breaks up into factors quadratic in t^2, of the form

$$(3)\quad T_1 = \alpha_1 t^4 + 2\beta_1 t^2 + \gamma_1, \qquad (4)\quad T_2 = -\alpha_2 t^4 - 2\beta_2 t^2 - \gamma_2,$$

and from Laguerre's formula (16) §7, when

$$(5)\qquad v_1 + v_2 = v = f\omega_3, \quad v_1 - v_2 = v' = f'\omega_3,$$

$$(6)\qquad \sqrt{e_3 - \wp v} = \sqrt{s_3 - \sigma} = \frac{1}{2}(\sqrt{\alpha_2\gamma_1} + \sqrt{\alpha_1\gamma_2}),$$

$$(7)\qquad \sqrt{e_3 - \wp v'} = \sqrt{s_3 - s} = \frac{1}{2}(\sqrt{\alpha_2\gamma_1} - \sqrt{\alpha_1\gamma_2}),$$

$$(8)\quad \sqrt{\alpha_2\gamma_1} = \sqrt{s_3 - \sigma} + \sqrt{s_3 - s}, \qquad (9)\quad \sqrt{\alpha_1\gamma_2} = \sqrt{s_3 - \sigma} - \sqrt{s_3 - s},$$

and

$$(10)\quad \alpha_1\alpha_2 = (L + L')^2, \qquad (11)\quad \gamma_1\gamma_2 = (L - L')^2,$$

$$(12)\qquad \sqrt{\alpha_1\alpha_2\gamma_1\gamma_2} = L^2 - L'^2 = s - \sigma = s(v') - s(v),$$

where σ or $s(v)$ was replaced by zero in §12 with μ odd.

Formula (13) §12 gives

$$(13)\quad \sqrt{4 . s_1 - \sigma . s_2 - \sigma . s_3 - \sigma} = \tfrac{1}{2}(\alpha_1\beta_2 + \alpha_2\beta_1)\sqrt{\gamma_1\gamma_2} + \tfrac{1}{2}(\beta_1\gamma_2 + \beta_2\gamma_1)\sqrt{\alpha_1\alpha_2}$$
$$= \tfrac{1}{2}(\sqrt{\alpha_2\gamma_1} + \sqrt{\alpha_1\gamma_2})(\beta_1\sqrt{\alpha_2\gamma_2} + \beta_2\sqrt{\alpha_1\gamma_1}),$$

so that, from (6),

$$(14)\qquad \sqrt{s_1 - \sigma . s_2 - \sigma} = \tfrac{1}{2}(\beta_1\sqrt{\alpha_2\gamma_2} + \beta_2\sqrt{\alpha_1\gamma_1}),$$

and similarly

$$(15)\qquad \sqrt{s_1 - s . s_2 - s} = \tfrac{1}{2}(\beta_1\sqrt{\alpha_2\gamma_2} - \beta_2\sqrt{\alpha_1\gamma_1}).$$

With one condition still at disposal we can make

$$(16)\qquad \sqrt{\alpha_1\gamma_1} = \sqrt{\alpha_2\gamma_2} = \sqrt{s - \sigma},$$

$$(17)\qquad \begin{matrix}\beta_1\\ \beta_2\end{matrix} = \frac{\sqrt{s_1 - \sigma . s_2 - \sigma} \pm \sqrt{s_1 - s . s_2 - s}}{\sqrt{s - \sigma}},$$

$$(18)\quad \begin{matrix}T_1\\ T_2\end{matrix} = \frac{\pm t^4 (L + L')(\sqrt{s_3 - \sigma} \mp \sqrt{s_3 - s}) \pm 2t^2(\sqrt{s_1 - \sigma . s_2 - \sigma} \pm \sqrt{s_1 - s . s_2 - s}) \pm (L - L')\sqrt{s_3 - \sigma} \pm \sqrt{s_3 - s}}{\sqrt{s - \sigma}}$$

suitable for $L^2 - L'^2 = s - \sigma$ positive; and then putting

$$t^2 \sqrt{\frac{L+L'}{L-L'}} = x, \tag{19}$$

$$\frac{T}{(L-L')^2} = \left[x^2 \frac{\sqrt{s_3-\sigma} - \sqrt{s_3-s}}{\sqrt{s-\sigma}} + 2x \frac{\sqrt{s_1-\sigma \,.\, s_2-\sigma} + \sqrt{s_1-s \,.\, s_2-s}}{s-\sigma} + \frac{\sqrt{s_3-\sigma} + \sqrt{s_3-s}}{\sqrt{s-\sigma}} \right] \tag{20}$$

$$\left[-x^2 \frac{\sqrt{s_3-\sigma} + \sqrt{s_3-s}}{\sqrt{s-\sigma}} - 2x \frac{\sqrt{s_1-\sigma \,.\, s_2-\sigma} - \sqrt{s_1-s \,.\, s_2-s}}{s-\sigma} - \frac{\sqrt{s_3-\sigma} - \sqrt{s_3-s}}{\sqrt{s-\sigma}} \right]$$

homogeneous in σ and s, and so independent of homogeneity factor.

An appropriate change of sign will serve for $L^2 - L'^2 = s - \sigma$ negative. The same form of T_1 and T_1 will hold for parameters

$$v, v' = \omega_1 + \frac{2r}{2n+1}\omega_3, \tag{21}$$

with an interchange of s_1 and s_3.

But with

$$v, v' = \omega_1 + \frac{2r+1}{2n+1}\omega_3, \tag{22}$$

s_2 and s_3 change place, and $\sqrt{s_2-\sigma}, \sqrt{s_2-s}$ are negative, so that we take

$$\frac{\alpha_1}{\alpha_2} = (L+L')\frac{\sqrt{s-s_2} \mp \sqrt{\sigma-s_2}}{\sqrt{s-\sigma}}, \tag{23}$$ $$\frac{\gamma_1}{\gamma_2} = -(L-L')\frac{\sqrt{s-s_2} \pm \sqrt{\sigma-s_2}}{\sqrt{s-\sigma}}, \tag{24}$$

$$\frac{\beta_1}{\beta_2} = -\frac{\sqrt{s_1-\sigma \,.\, \sigma-s_3} \pm \sqrt{s_1-s \,.\, s-s_3}}{\sqrt{s-\sigma}}. \tag{25}$$

Finally, with

$$v, v' = \omega_1 + \frac{2r+1}{2n}\omega_3, \tag{26}$$

s_2 and s_3 change place again in (23), (24), (25).

The discriminant of T_1 in (3), (18) as a quadratic in t^2

$$\beta_1^2 - \alpha_1\gamma_1 = \frac{(\sqrt{s_1-\sigma \,.\, s_2-s} + \sqrt{s_1-s \,.\, s_2-\sigma})^2}{s-\sigma}, \tag{27}$$

so that the roots can be expressed rationally by $\sqrt{s_a-\sigma}, \sqrt{s_a-s}$.

When the shape of the body is such that $p = p'$, the quartic Z reduces to a cubic in z, and our equations connecting $z = \cos\vartheta$ and ψ become the same as

those required for the motion of a symmetrical top (*Annals of Mathematics*, vol. 5, 1904); in this case t^2+1 is a factor of T, say of T_1, so that

(28) $$\alpha_1 - 2\beta_1 + \gamma_1 = 0,$$

reducing in the case of $v = \omega_1 + f\omega_3$ to

(29) $$\left\{\begin{aligned}&(L+L')(\sqrt{s-s_2}-\sqrt{\sigma-s_2})-(L-L')(\sqrt{s-s_2}+\sqrt{\sigma-s_2})\\ &\qquad = 2(\sqrt{s_1-\sigma\,.\,\sigma-s_3}+\sqrt{s_1-s\,.\,s-s_3}),\end{aligned}\right.$$

and then

(30) $$\left\{\begin{aligned}&(L+L')(\sqrt{s-s_2}-\sqrt{\sigma-s_2})+(L-L')(\sqrt{s-s_2}+\sqrt{\sigma-s_2})\\ &\qquad = 2(\sqrt{s_1-\sigma\,.\,s-s_3}+\sqrt{s_1-s\,.\,\sigma-s_3}),\end{aligned}\right.$$

whence L and L' in a case where the motion is the same as top motion.

We can put

(31) $$L+L' = \sqrt{s-\sigma}\,.\,a, \quad L-L' = \sqrt{s-\sigma}\,.\,\frac{1}{a},$$

and then from (29), (30),

(32) $$a = \frac{(\sqrt{s_1-\sigma}+\sqrt{s_1-s})(\sqrt{s-s_3}+\sqrt{\sigma-s_3})}{\sqrt{s-\sigma}(\sqrt{s-s_2}-\sqrt{\sigma-s_2})},$$

while

(33) $$M^2 = s\tfrac{1}{2}(v+v') - s\tfrac{1}{2}(v-v'),$$

denoting σ and s by $s(v)$ and $s(v')$.

18. Before proceeding further, test by a simple case

(1) $$v = \tfrac{1}{2}\omega_3,$$

so that we can take (*Phil. Trans.*, §37),

(2) $$s_1 = \frac{1}{\varkappa}, \qquad s_2 = \varkappa, \quad s_3 = 0, \quad \sigma = -1,$$

(3) $$P = \tfrac{1}{2}\left(\frac{1}{o}+o\right) = \frac{\text{zs}\,\tfrac{1}{2}K'}{o}, \quad \text{with } \varkappa = o^2,$$

and now put

(4) $$s_3 - s = b^2, \qquad s_1 - s = \frac{1}{o^2} + b^2, \qquad s_2 - s = o^2 + b^2,$$

(5) $$B^2 = s_1 - s\,.\,s_2 - s = 1 + (4P^2-2)b^2 + b^4 = (1-b^2)^2 + 4P^2b^2,$$

(6) $$4P^2 = s_1 - \sigma\,.\,s_2 - \sigma,$$

(7) $$L^2 - L'^2 = s - \sigma = 1 - b^2.$$

Try

(8) $$\psi' = \tfrac{1}{2}\cos^{-1}\frac{t^2+K_1}{Nt^2}, \quad \sqrt{T_1} = \tfrac{1}{2}\sin^{-1}\frac{t^2+K_2}{Nt^2}\sqrt{T_2},$$

with

(9) $$\alpha_1 = \alpha_2 = (L+L')^2,$$ (10) $$\gamma_1 = (1+b)^2, \quad \gamma_2 = (1-b)^2,$$

(11) $$\beta_1 = \frac{(2P+B)(1+b)}{L-L'},$$ (12) $$\beta_2 = \frac{(2P-B)(1-b)}{L-L'},$$

and then to satisfy the differentiation

(13) $$\frac{d\psi'}{dt} = -\frac{(L+L')^2t^4 + 2(L+L')Pt^2 + L^2 - L'^2}{t\sqrt{T_1T_2}},$$

we find

(14) $$K_1 = -\frac{2Pb+B}{(L+L')(1+b)}, \quad K_2 = \frac{2Pb+B}{(L+L')(1-b)}.$$

The results can be thrown into a more symmetrical shape by putting

(15) $$t^2 = x\sqrt{\frac{L-L'}{L+L'}}\left(\text{or } x\sqrt{\frac{L'-L}{L'+L}} \text{ if requisite}\right),$$

and writing

(16) $$\begin{cases} \psi' = \tfrac{1}{2}\cos^{-1}\dfrac{x\sqrt{\dfrac{-2Pb+B}{1-b}} - \sqrt{\dfrac{2Pb+B}{1+b}}}{Nx}\sqrt{X_1} \\[2ex] \quad = \tfrac{1}{2}\sin^{-1}\dfrac{x\sqrt{\dfrac{-2Pb+B}{1+b}} + \sqrt{\dfrac{2Pb+B}{1-b}}}{Nx}\sqrt{X_2}, \end{cases}$$

(17) $$X_1 = \quad x^2\sqrt{\frac{1-b}{1+b}} + 2x\frac{2P+B}{1-b^2} + \sqrt{\frac{1+b}{1-b}},$$

(18) $$X_2 = -x^2\sqrt{\frac{1+b}{1-b}} - 2x\frac{2P-B}{1-b^2} - \sqrt{\frac{1-b}{1+b}},$$

leading to

(19) $$\frac{d\psi'}{dx} = -\frac{x^2 + 2x\dfrac{P}{\sqrt{(1-b^2)}} + 1}{2x\sqrt{X_1X_2}},$$

and, squaring and adding, we find

(20) $$N^2 = -\frac{16P}{(1-b^2)^{\frac{3}{2}}},$$

implying that X_1 and X_2 should change sign to obtain a real result.

If $L^2 - L'^2$, $s - \sigma$, and $1 - b^2$ are negative, so that it is requisite to take

(21) $$t^2 = x\sqrt{\frac{L' - L}{L' + L}},$$

then we write

(22) $$\left\{\begin{aligned} \psi' &= \tfrac{1}{2}\cos^{-1}\frac{x\sqrt{\dfrac{B-2Pb}{b-1}} - \sqrt{\dfrac{B+2Pb}{b+1}}}{Nx}\sqrt{X_1} \\ &= \tfrac{1}{2}\sin^{-1}\frac{x\sqrt{\dfrac{B-2Pb}{b+1}} - \sqrt{\dfrac{B+2Pb}{b-1}}}{Nx}\sqrt{X_2}, \end{aligned}\right.$$

(23) $$X_1 = \quad x^2\sqrt{\frac{b-1}{b+1}} - 2x\frac{B+2P}{b^2-1} + \sqrt{\frac{b+1}{b-1}},$$

(24) $$X_2 = -x^2\sqrt{\frac{b+1}{b-1}} + 2x\frac{B-2P}{b^2-1} - \sqrt{\frac{b-1}{b+1}},$$

leading to

(25) $$\frac{d\psi'}{dx} = -\frac{x^2 + 2x\dfrac{P}{\sqrt{(b^2-1)}} - 1}{2x\sqrt{X_1X_2}},$$

and

(26) $$N^2 = -\frac{16P}{(b^2-1)^{\frac{3}{2}}}.$$

With a parameter

(27) $$v = \omega_1 + \tfrac{1}{2}\omega_3,$$

$s_3 - \sigma$ and $s_3 - s$ are negative, and we take

(28) $$\sigma = 1, \quad \sigma - s_3 = 1, \quad s - s_3 = b^2,$$

(29) $$L^2 - L'^2 = s - \sigma = b^2 - 1,$$

(30) $$B^2 = s_1 - s \,.\, s - s_2 = \left(\frac{1}{o^2} - b^2\right)(b^2 - o^2) = 4P^2b^2 - (b^2-1)^2,$$

(31) $$4P^2 = s_1 - \sigma \,.\, \sigma - s_2,$$

(32) $$P = \tfrac{1}{2}\left(\frac{1}{o} - o\right) = \frac{\operatorname{zn}\tfrac{1}{2}K'}{o}, \quad x = o^2.$$

Now, with

$$(33) \qquad t^2 = x\sqrt{\frac{L - L'}{L + L'}},$$

$$(34) \quad \left\{ \begin{aligned} \psi' &= \tfrac{1}{2}\cos^{-1} \frac{x\,\dfrac{\sqrt{b^2+2Pb-1}+\sqrt{-b^2+2Pb+1}}{\sqrt{b-1}} - \dfrac{\sqrt{b^2+2Pb-1}-\sqrt{-b^2+2Pb+1}}{\sqrt{b+1}}}{Nx}\sqrt{X_1} \\ &= \tfrac{1}{2}\sin^{-1} \frac{x\,\dfrac{\sqrt{b^2+2Pb-1}+\sqrt{-b^2+2Pb+1}}{\sqrt{b+1}} - \dfrac{\sqrt{b^2+2Pb-1}-\sqrt{-b^2+2Pb+1}}{\sqrt{b-1}}}{Nx}\sqrt{X_2}, \end{aligned} \right.$$

$$(35) \qquad X_1 = x^2\sqrt{\frac{b-1}{b+1}} + 2x\,\frac{B-2P}{b^2-1} - \sqrt{\frac{b+1}{b-1}},$$

$$(36) \qquad X_2 = -x^2\sqrt{\frac{b+1}{b-1}} - 2x\,\frac{B+2P}{b^2-1} + \sqrt{\frac{b-1}{b+1}},$$

$$(37) \qquad N^2 = -\frac{32P}{(b^2-1)^{\frac{3}{2}}},$$

leading on differentiation to

$$(38) \qquad \frac{d\psi'}{dx} = -\frac{x^2 + 2x\,\dfrac{P}{\sqrt{(b^2-1)}} + 1}{2x\sqrt{X_1X_2}}.$$

When $L^2 - L'^2 = s - \sigma$ is negative, so that we must take

$$(39) \qquad t^2 = x\sqrt{\frac{L' - L}{L' + L}},$$

an appropriate change of sign must be made equivalent to replacing x by xi in (16) where $v = \frac{1}{2}\omega_3$, so that

$$(40) \quad \left\{ \begin{aligned} \psi' &= \tfrac{1}{2}\cos^{-1}\left(x\sqrt{\frac{B-2Pb}{b-1}} + \sqrt{\frac{B+2Pb}{b+1}}\right)\sqrt{x^2\sqrt{\frac{b-1}{b+1}} + 2x\,\frac{B+2P}{b^2-1} + \sqrt{\frac{b+1}{b-1}}} \div Nx \\ &= \tfrac{1}{2}\sin^{-1}\left(x\sqrt{\frac{B-2Pb}{b+1}} + \sqrt{\frac{B+2Pb}{b-1}}\right)\sqrt{-x^2\sqrt{\frac{b+1}{b-1}} - 2x\,\frac{B-2P}{b^2-1} - \sqrt{\frac{b-1}{b+1}}} \div Nx, \end{aligned} \right.$$

$$(41) \qquad N^2 = -\frac{16P}{(b^2-1)^{\frac{3}{2}}},$$

and

$$(42) \qquad \frac{d\psi'}{dx} = -\frac{x^2 - 2x\,\dfrac{P}{\sqrt{b^2-1}} - 1}{2x\sqrt{X_1X_2}}.$$

So also, replacing x by xi in (34) where $v = \omega_1 + \frac{1}{2}\omega_3$,

$$(43)\quad \begin{cases} \psi' = \frac{1}{2}\cos^{-1}\left(x\,\dfrac{\sqrt{1+2Pb-b^2}+\sqrt{-1+2Pb+b^2}}{\sqrt{1-b}} + \dfrac{\sqrt{1+2Pb-b^2}-\sqrt{-1+2Pb+b^2}}{\sqrt{1+b}}\right)\sqrt{X_1} \div Nx \\ \quad = \frac{1}{2}\sin^{-1}\left(x\,\dfrac{\sqrt{1+2Pb-b^2}+\sqrt{-1-2Pb+b^2}}{\sqrt{1+b}} - \dfrac{\sqrt{1+2Pb-b^2}-\sqrt{-1+2Pb+b^2}}{\sqrt{1-b}}\right)\sqrt{X_2} \div Nx, \end{cases}$$

$$(44)\quad X_1 = \quad x^2\sqrt{\frac{1-b}{1+b}} - 2x\,\frac{2P-B}{1-b^2} - \sqrt{\frac{1+b}{1-b}},$$

$$(45)\quad X_2 = -x^2\sqrt{\frac{1+b}{1-b}} + 2x\,\frac{2P+B}{1-b^2} + \sqrt{\frac{1-b}{1+b}},$$

$$(46)\quad N^2 = -\frac{32P}{(1-b^2)^{\frac{3}{2}}},$$

so that X_1 and X_2 must change sign for a real result, when P is positive.

19. Generally with $\mu = 4n$, and a parameter

$$(1)\qquad v = \frac{\omega_3}{2n},$$

and beginning with the case of

$$(2)\qquad L'^2 - L^2 = \sigma - s \text{ positive},$$

we put

$$(3)\qquad t^2 = x\sqrt{\frac{L'-L}{L'+L}},$$

and can satisfy

$$(4)\qquad \frac{d\psi'}{dx} = -\frac{x^2 + 2x\dfrac{P}{\sqrt{\sigma - s}} - 1}{2x\sqrt{X_1 X_2}},$$

by

$$(5)\quad \begin{cases} \psi' = \dfrac{1}{2n}\cos^{-1}(A_1x^{2n-1} + B_1x^{2n-2} + \ldots + K_1x + L_1)\sqrt{(X_1)} \div Nx^n \\ \quad = \dfrac{1}{2n}\sin^{-1}(A_2x^{2n-1} + B_2x^{2n-2} + \ldots + K_2x + L_2)\sqrt{(X_2)} \div Nx^n, \end{cases}$$

where

$$(6)\qquad X_1 = \quad x^2\,\frac{\sqrt{s_3-s}-\sqrt{s_3-\sigma}}{\sqrt{\sigma-s}} - 2x\,\frac{\sqrt{s_1-s\,.\,s_2-s}+\sqrt{s_1-\sigma\,.\,s_2-\sigma}}{\sigma-s} + \frac{\sqrt{s_3-s}+\sqrt{s_3-\sigma}}{\sqrt{\sigma-s}},$$

$$(7)\qquad X_2 = -x^2\,\frac{\sqrt{s_3-s}+\sqrt{s_3-\sigma}}{\sqrt{\sigma-s}} + 2x\,\frac{\sqrt{s_1-s\,.\,s_2-s}-\sqrt{s_1-\sigma\,.\,s_2-\sigma}}{\sigma-s} - \frac{\sqrt{s_3-s}-\sqrt{s_3-\sigma}}{\sqrt{\sigma-s}}.$$

To simplify further we put

$$(8)\qquad s_3 - s = b^2, \qquad\qquad s_3 - \sigma = c^2,$$

$$(9)\qquad (s_1 - s)(s_2 - s) = (s_1 - s_3 + b^2)(s_2 - s_3 + b^2) = B,$$

suppose; and this can be normalized to

(10) $$B = \left(\frac{1}{o^2} + b^2\right)(o^2 + b^2),$$

where o denotes the octahedron irrationality, defined by

(11) $$o^4 = x^2 = \frac{s_2 - s_3}{s_1 - s_3};$$

and with

(12) $$C = \left(\frac{1}{o^2} + c^2\right)(o^2 + c^2),$$

(13) $$X_1 = \quad x^2\sqrt{\frac{b-c}{b+c}} - 2x\frac{\sqrt{B}+\sqrt{C}}{b^2-c^2} + \sqrt{\frac{b+c}{b-c}},$$

(14) $$X_2 = -x^2\sqrt{\frac{b+c}{b-c}} + 2x\frac{\sqrt{B}-\sqrt{C}}{b^2-c^2} - \sqrt{\frac{b-c}{b+c}}.$$

But with

(15) $$L^2 - L'^2 = s - \sigma \text{ positive},$$

we put

(16) $$t^2 = x\sqrt{\frac{L-L'}{L+L'}};$$

b and c must change place, and

(17) $$\frac{d\psi'}{dx} = -\frac{x^2 + 2x\dfrac{P}{\sqrt{s-\sigma}} + 1}{2x\sqrt{X_1X_2}}.$$

With a parameter

(18) $$v = \omega_1 + \frac{\omega_3}{2n},$$

we have to put

(19) $$s - s_3 = b^2, \qquad \sigma - s_3 = c^2,$$

(20) $$B = s_1 - s\,.\,s - s_2 = (s_1 - s_3 - b^2)(b^2 - s_2 + s_3),$$

or normalized

(21) $$B = \left(\frac{1}{o^2} - b^2\right)(b^2 - o^2),$$

(22) $$C = \left(\frac{1}{o^2} - c^2\right)(c^2 - o^2),$$

and then

(23) $$X_1 = \quad x^2\sqrt{\frac{b-c}{b+c}} + 2x\frac{\sqrt{B}-\sqrt{C}}{b^2-c^2} - \sqrt{\frac{b+c}{b-c}},$$

(24) $$X_2 = -x^2\sqrt{\frac{b+c}{b-c}} - 2x\frac{\sqrt{B}+\sqrt{C}}{b^2-c^2} + \sqrt{\frac{b-c}{b+c}},$$

or else b and c, B and C change place, according as we have to put

(25) $$t^2 = x\sqrt{\frac{L-L'}{L+L'}}, \quad \text{or } x\sqrt{\frac{L'-L}{L'+L}}.$$

In the determination of the coefficients $A_1, \ldots. L_1, A_2, \ldots., L_2$, we can take

(26) $$A_1 = \frac{1}{\lambda}\sqrt[4]{\frac{b+c}{b-c}}, \quad L_1 = -\lambda\sqrt[4]{\frac{b-c}{b+c}},$$

(27) $$A_2 = \frac{1}{\lambda}\sqrt[4]{\frac{b-c}{b+c}}, \quad L_2 = -\lambda\sqrt[4]{\frac{b+c}{b-c}};$$

(28) $$A_1 L_1 = A_2 L_2 = -1,$$

and the value of λ may be inferred by induction from the associated pseudo-elliptic integral discussed in the *Phil. Trans.* 1904

(29) $$\begin{cases} I(v) = \displaystyle\int \frac{P(b^2 \sim c^2) + Q}{b^2 \sim c^2} \frac{db}{\sqrt{B}} \\ \quad = \dfrac{1}{n} \log \dfrac{E_1\sqrt{\frac{1}{2}B_1'} + E_2\sqrt{\frac{1}{2}B_2'}}{(b^2 \sim c^2)^{\frac{1}{2}n}}, \quad B = B_1' B_2'; \end{cases}$$

from which we may take

(30) $$\lambda = e^{nI(v)} = \frac{E_1\sqrt{\frac{1}{2}B_1'} + E_2\sqrt{\frac{1}{2}B_2'}}{(b^2 \sim c^2)^{\frac{1}{2}n}}.$$

The remaining coefficients are now determined readily from the differentiation in (17).

Tested by $\mu = 8$ (*Phil. Trans.*, §38),

(31) $$v = \omega_1 + \tfrac{1}{4}\omega_3,$$

(32) $$I(v) = \tfrac{1}{2}\,\mathrm{ch}^{-1}\frac{b-c^2}{b^2-c^2}\sqrt{\tfrac{1}{2}B_1'} = \tfrac{1}{2}\,\mathrm{sh}^{-1}\frac{b+c^2}{b^2-c^2}\sqrt{\tfrac{1}{2}B_2'},$$

(33) $$\begin{matrix}B_1' \\ B_2'\end{matrix} = \pm(b^2+1) + \left(\frac{1}{o} + o\right)b,$$

(34) $$\tfrac{1}{2}\left(\frac{1}{o} + o\right) = \frac{\mathrm{zs}\frac{1}{2}K'}{o} = \left(\frac{c^2+1}{2c}\right)^2,$$

(35) $$P = \frac{\mathrm{zn}\frac{1}{4}K'}{o} = \frac{(3c^2+1)(c^2-1)}{8c^2},$$

(36) $$Q = \tfrac{1}{2}(c^4 - 1) = c\sqrt{C};$$

and working with

(37) $$t^2 = x\sqrt{\frac{L-L'}{L+L'}}, \quad L^2 - L'^2 = s - \sigma \text{ positive},$$

then

$$(38)\qquad \frac{d\psi'}{dx} = -\frac{x^2 + 2x\dfrac{P}{\sqrt{b^2-c^2}} + 1}{2x\sqrt{X_1X_2}},$$

is satisfied by

$$(39)\qquad \begin{cases} \psi' = \frac{1}{4}\cos^{-1}(A_1x^3 + B_1x^2 + K_1x + L_1)\sqrt{X_1} \div Nx^2 \\ = \frac{1}{4}\sin^{-1}(A_2x^3 + B_2x^2 + K_2x + L_2)\sqrt{X_2} \div Nx^2, \end{cases}$$

with X_1 and X_2 as in (23) and (24),

$$(40)\qquad \lambda = e^{2I} = \frac{(b^2-c^2)\sqrt{\frac{1}{2}B_1'} + (b^2+c^2)\sqrt{\frac{1}{2}B_2'}}{b^2-c^2},$$

and in addition to the values of A_1, L_1, A_2, L_2, we find

$$(41)\qquad \frac{B_1}{A_1} = \left(\frac{\sqrt{B}+\sqrt{C}}{b-c} + 2\frac{\sqrt{B}-\sqrt{C}}{b+c} - 4P\right) \div \sqrt{b^2-c^2},$$

$$(42)\qquad \frac{B_2}{A_2} = \left(2\frac{\sqrt{B}+\sqrt{C}}{b-c} + \frac{\sqrt{B}-\sqrt{C}}{b+c} - 4P\right) \div \sqrt{b^2-c^2},$$

$$(43)\qquad \frac{K_1}{L_1} = \left(-\frac{\sqrt{B}+\sqrt{C}}{b+c} - 2\frac{\sqrt{B}-\sqrt{C}}{b-c} - 4P\right) \div \sqrt{b^2-c^2},$$

$$(44)\qquad \frac{K_2}{L_2} = \left(-2\frac{\sqrt{B}+\sqrt{C}}{b+c} - \frac{\sqrt{B}-\sqrt{C}}{b-c} - 4P\right) \div \sqrt{b^2-c^2}.$$

$$(45)\qquad N^2 = -4\frac{(c^2+1)(c^4-1)^2}{(b^2-c^2)^4c},$$

Tested numerically with

$$(46)\qquad b = \frac{\sqrt{5}+1}{2},\quad c = \frac{\sqrt{5}-1}{2},$$

we find

$$(47)\qquad \begin{cases} \psi' = \frac{1}{4}\cos^{-1}\left(\dfrac{x^3}{\sqrt[4]{5}\sqrt[8]{5}} + x^2\dfrac{\sqrt{5}+1}{\sqrt[8]{5}} + x\dfrac{\sqrt{5}+1}{2}\sqrt[8]{5} - \sqrt[4]{5}\sqrt[8]{5}\right) \\ \qquad\qquad \sqrt{\left(x^2\sqrt[4]{5} + x - \dfrac{1}{\sqrt[4]{5}}\right)} \div Nx^2 \\ = \frac{1}{4}\sin^{-1}\left(\dfrac{x^3}{\sqrt[8]{5}} + x^2\dfrac{3+\sqrt{5}}{\sqrt[4]{5}\sqrt[8]{5}} + x\dfrac{\sqrt{5}+1}{2\sqrt[8]{5}} - \sqrt[8]{5}\right) \\ \qquad\qquad \sqrt{\left(-\dfrac{x^2}{\sqrt[4]{5}} - x\sqrt{5} + \sqrt[4]{5}\right)} \div Nx^2, \end{cases}$$

leads to the requisite differentiation (38) with

$$(48)\qquad \frac{P}{\sqrt{(b^2-c^2)}} = \frac{1-2\sqrt{5}}{8\sqrt[4]{5}},\quad N = \frac{3-\sqrt{5}}{\sqrt[4]{5}}.$$

In this case, and also generally when $bc = 1$,

(49) $$v' = \omega_1 - \tfrac{3}{4}\omega_3, \quad v_1 = \omega_1 - \tfrac{1}{4}\omega_3, \quad v_2 = \tfrac{1}{2}\omega_3,$$

so that $\alpha_1, \beta_1, \gamma_1, \delta_1$, can be expressed by algebraical functions.

With $\mu = 12$ (*Phil. Trans.*, §40), $v = \omega_1 + \tfrac{1}{6}\omega_3$,

(50) $c^2 = a + a^2 + a^3$,

(51) $\begin{matrix} B_1' \\ B_2' \end{matrix} = \pm b^2 + \tfrac{1}{2}(1+a)^3(1-a)b \mp a^3(1+a+a^2)$,

(52) $\sqrt{C} = \tfrac{1}{2}(1-a^4)(a+a^2+a^3)$,

(53) $P = \tfrac{1}{12}(1-a)(5+3a+3a^2+a^3)$,

(54) $$\lambda = \frac{[b^2+(1-a)b-1-a-a^2]\sqrt{\tfrac{1}{2}B_1'}+[b^2-(1-a)b-1-a-a^2]\sqrt{\tfrac{1}{2}B_2'}}{(b^2-a-a^2-a^3)^{\frac{3}{2}}},$$

and thence the calculation can be completed.

Material for the construction of cases corresponding to parameter

(55) $$v = \omega_1 + \tfrac{1}{8}\omega_3, \quad \omega_1 + \tfrac{1}{10}\omega_3, \quad \ldots\ldots\ldots,$$

will be found in *Phil. Trans.* 1904.

20. With $\mu = 4n+2$, the method of reduction given in §24, *Phil. Trans.*, p. 251, must be followed; choosing a parameter

(1) $$v = \omega_1 + \frac{2r+1}{2n+1}\omega_3,$$

and using the new variable

(2) $$x = t^2\sqrt{\frac{L'+L}{L'-L}},$$

implying that $L^2 - L'^2$ and $s - \sigma$ are negative, we can write

(3) $$\begin{cases} \psi' = \dfrac{1}{2n+1}\cos^{-1}\dfrac{A_1x^{2n}+B_1x^{2n-1}+\ldots+K_1x+L_1}{Nx^{n+\frac{1}{2}}}\sqrt{X_1} \\ \quad = \dfrac{1}{2n+1}\sin^{-1}\dfrac{A_2x^{2n}+B_2x^{2n-1}+\ldots+K_2x+L_2}{Nx^{n+\frac{1}{2}}}\sqrt{X_2}, \end{cases}$$

satisfying the differentiation

(4) $$\frac{d\psi'}{dx} = -\frac{x^2+2x\dfrac{P}{\sqrt{(\sigma-s)}}-1}{2x\sqrt{X_1X_2}},$$

and

(5) $$X_1 = \alpha_1 x^2 + 2\beta_1 x + \gamma_1, \quad X_2 = -\alpha_2 x^2 - 2\beta_2 x - \gamma_2,$$

(6) $$\alpha_1 = \frac{\sqrt{\sigma - s_2} - \sqrt{s - s_2}}{\sqrt{\sigma - s}} = -\gamma_2,$$

(7) $$\alpha_2 = \frac{\sqrt{\sigma - s_2} + \sqrt{s - s_2}}{\sqrt{\sigma - s}} = -\gamma_1,$$

(8) $$\alpha_1 \alpha_2 = \gamma_1 \gamma_2 = 1,$$

(9) $$\begin{matrix}\beta_1 \\ \beta_2\end{matrix} = \frac{\sqrt{s_1 - \sigma \cdot \sigma - s_3} \pm \sqrt{s_1 - s \cdot s - s_3}}{\sigma - s}.$$

Then

(10) $$A_1 = \frac{1}{\lambda}\sqrt{\alpha_2}, \quad L_1 = +\lambda\sqrt{\alpha_1},$$

(11) $$A_2 = \frac{1}{\lambda}\sqrt{\alpha_1}, \quad L_2 = -\lambda\sqrt{\alpha_2};$$

and guided by induction we find

(12) $$\lambda^2, \frac{1}{\lambda^2} = \frac{E\sqrt{s_1 - s \cdot s - s_3} \pm F\sqrt{s - s_2}}{(\sigma - s)^{n+\frac{1}{2}}},$$

derivable from the associated pseudo-elliptic integral

(13) $$\begin{cases} I(v) = \int_{s_2}^{s} \frac{P(\sigma - s) - Q}{\sigma - s} \frac{ds}{\sqrt{(-S)}} \\ \quad = \frac{1}{2n+1} \operatorname{ch}^{-1} \frac{E\sqrt{s_1 - s \cdot s - s_3}}{(\sigma - s)^{n+\frac{1}{2}}} = \frac{1}{2n+1} \operatorname{sh}^{-1} \frac{F\sqrt{s - s_2}}{(\sigma - s)^{n+\frac{1}{2}}}, \end{cases}$$

so that

(14) $$\lambda, \frac{1}{\lambda} = e^{\pm(n+\frac{1}{2})I(v)}$$

Performing the differentiation of the $\cos^{-1}$ and $\sin^{-1}$ in (3),

(15) $$\begin{cases} \frac{d\psi'}{dx} = -\frac{x^2 + 2x\frac{P}{\sqrt{\sigma - s}} - 1}{2x\sqrt{X_1 X_2}} \\ \quad = -\frac{N_1}{2(A_2 x^{2n} + \dots + L_2)x\sqrt{X_1 X_2}} = \frac{N_2}{2(A_1 x^{2n} + \dots + L_1)x\sqrt{X_1 X_2}}, \end{cases}$$

and

$$(16)\quad \begin{cases} N_1 = [(2n-1)A_1x^{2n} + (2n-3)B_1x^{2n-1} + \ldots - (2n-1)K_1x - (2n+1)L_1](a_1x^2 + 2\beta_1x + \gamma_1) \\ \quad + (A_1x^{2n} + B_1x^{2n-1} + \ldots + K_1x + L_1)(2a_1x^2 + 2\beta_1x + 0) \\ \quad = (A_2x^{2n} + B_2x^{2n-1} + \ldots + K_2x + L_2)\left[(2n+1)x^2 + 2x\dfrac{(2n+1)P}{\sqrt{\sigma - s}} - 2n - 1\right], \end{cases}$$

$$(17)\quad \begin{cases} N_2 = [(2n-1)A_2x^{2n} + (2n-3)B_2x^{2n-1} + \ldots - (2n-1)K_2x - (2n+1)L_2](-a_2x^2 - 2\beta_2x - \gamma_2) \\ \quad + (A_2x^{2n} + Bx^{2n-1} + \ldots + K_2x + L_2)(-2a_2x^2 - 2\beta_2x - 0) \\ \quad = (A_1x^{2n} + B_1x^{2n-1} + \ldots + K_1x + L_1)\left[-(2n+1)x^2 - 2x\dfrac{(2n+1)P}{\sqrt{\sigma - s}} + 2n + 1\right], \end{cases}$$

and more than sufficient equations are obtained by equating cofficients in (16) and (17) for the determination of the $A, B, \ldots, K, L$, and for their verification, when s, s_1, s_2, s_3, σ and P are assigned for a given integer n.

Thus we find the values of A_1, A_2 and L_1, L_2 given above; and further

$$(18)\qquad \frac{B_1}{A_1} = (n - \tfrac{1}{2})\frac{\beta_1}{\alpha_1} + (n + \tfrac{1}{2})\frac{\beta_2}{\alpha_2} - \frac{(2n+1)P}{\sqrt{\sigma - s}},$$

$$(19)\qquad \frac{B_2}{A_2} = (n + \tfrac{1}{2})\frac{\beta_1}{\alpha_1} + (n - \tfrac{1}{2})\frac{\beta_2}{\alpha_2} - \frac{(2n+1)P}{\sqrt{\sigma - s}},$$

$$(20)\qquad \frac{K_1}{L_1} = -(n + \tfrac{1}{2})\frac{\beta_1}{\alpha_2} - (n - \tfrac{1}{2})\frac{\beta_2}{\alpha_1} + \frac{(2n+1)P}{\sqrt{\sigma - s}},$$

$$(21)\qquad \frac{K_2}{L_2} = -(n - \tfrac{1}{2})\frac{\beta_1}{\alpha_2} - (n + \tfrac{1}{2})\frac{\beta_2}{\alpha_1} + \frac{(2n+1)P}{\sqrt{\sigma - s}};$$

and so on.

If in addition

$$(22)\qquad v' = \omega_2, \quad s = s_2,$$

$$(23)\qquad X_1 = x^2 + 2x\frac{\sqrt{s_1 - \sigma \cdot \sigma - s_3} + \sqrt{s_1 - s_2 \cdot s_2 - s_3}}{\sigma - s_2} - 1,$$

$$(24)\qquad X_2 = -x^2 + 2x\frac{-\sqrt{s_1 - \sigma \cdot \sigma - s_3} + \sqrt{s_1 - s_2 \cdot s_2 - s_3}}{\sigma - s_2} + 1$$

and we can take

$$(25)\qquad A_1 = A_2 = 1, \quad L_1 = -L_2 = 1.$$

Thus, for example, utilising subsequent results in §22, we have the following cases; for $\mu = 6$,

$$(26)\quad \begin{cases} \psi' = \tfrac{1}{3}\cos^{-1}(x^2 + 0 + 1)\sqrt{\left(x^2 + 2x\dfrac{a+1}{\sqrt{2a-1}} - 1\right)} \div Nx^{\frac{3}{2}}, \\[2ex] \quad = \tfrac{1}{3}\sin^{-1}\left(x^2 + \dfrac{2x}{\sqrt{2a-1}} - 1\right)\sqrt{\left(x^2 - 2x\dfrac{a-1}{\sqrt{2a-1}} + 1\right)} \div Nx^{\frac{3}{2}}, \end{cases}$$

$$(27)\quad N = \frac{4a}{(2a-1)^{\frac{3}{4}}},$$

$$(28)\quad \frac{d\psi'}{dx} = -\frac{x^2 + 2x\dfrac{\sqrt{2a-1}}{3} - 1}{2x\sqrt{\left[-x^4 - \dfrac{4ax^3}{\sqrt{2a-1}} - 2x^2\dfrac{2a^2-2a+1}{2a-1} + \dfrac{4ax}{\sqrt{2a-1}} - 1\right]}},$$

and putting

$$(29)\quad x - \frac{1}{x} = y,$$

$$(30)\quad \begin{cases} \psi' = \tfrac{1}{3}\cos^{-1}\sqrt{(y^2+1)}\sqrt{\left(y + \dfrac{a+1}{\sqrt{2a-1}}\right)} \div N' \\[2ex] \quad = \tfrac{1}{3}\sin^{-1}\left(y + \dfrac{1}{\sqrt{2a-1}}\right)\sqrt{\left(-y - \dfrac{a-1}{\sqrt{2a-1}}\right)} \div N'. \end{cases}$$

For $\mu = 10$,

$$(31)\quad \begin{cases} \psi' = \tfrac{1}{5}\cos^{-1}\left[x^7 + 4x^3\dfrac{c-1}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1} - 2x^2\dfrac{(c-1)(c-3)}{-c^2+4c+1} - 4x\dfrac{c-1}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1} + 1\right] \\[2ex] \qquad\qquad \sqrt{\left[x^2 + 2x\dfrac{c^3-c^2+3c+1}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1} + 1\right]} \div Nx^{\frac{5}{2}} \\[2ex] \quad = \tfrac{1}{5}\sin^{-1}\left[x^2 + 2x\dfrac{(c-1)(c^2+1)}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1} - 1\right](x^2+1) \\[2ex] \qquad\qquad \sqrt{\left[-x^2 + 2x\dfrac{c^3-c^2-5c+1}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1} + 1\right]} \div Nx^{\frac{5}{2}}, \end{cases}$$

or, putting

$$x - \frac{1}{x} = 2y,$$

$$(32)\quad \begin{cases} \psi' = \tfrac{1}{5}\cos^{-1}\dfrac{1}{N'}\left[y^2 + 2y\dfrac{c-1}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1} - \dfrac{c^2-4c+1}{-c^2+4c+1}\right] \\[2ex] \qquad\qquad \sqrt{\left[y + \dfrac{c^3-c^2+3c+1}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1}\right]} \\[2ex] \quad = \tfrac{1}{5}\sin^{-1}\dfrac{1}{N'}\left[y + \dfrac{(c-1)(c^2+1)}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1}\right]\sqrt{(y^2+1)} \\[2ex] \qquad\qquad \sqrt{\left[-y + \dfrac{c^3-c^2-5c+1}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1}\right]}, \end{cases}$$

$$(33)\quad N'^2 = \frac{128c^4}{(c+1)^4(-c^2+4c+1)^2\sqrt{(c^2-1)(-c^2+4c+1)}},$$

leading on differentiation to

$$(34)\quad \frac{d\psi'}{dy} = \frac{y + \dfrac{\frac{1}{5}(c+3)(-c^2+4c+1)}{(c+1)\sqrt{(c^2-1)(-c^2+4c+1)}}}{2\sqrt{y^2+1}\sqrt{y + \dfrac{c^3-c^2+3c+1}{(c+1)\sqrt{(c^2-1)(-c^2+4c+1)}}}\sqrt{-y + \dfrac{c^3-c^2-5c+1}{(c+1)\sqrt{(c^2-1)(-c^2+4c+1)}}}}$$

Utilizing the results in *Phil. Trans.*, §§31, 32, we could go on to the construction of the case corresponding to $\mu = 14$,

$$(35)\quad \begin{cases} \psi' = \frac{1}{7}\cos^{-1} C(v^2 + E_1 v + E_2)\sqrt{v^2+1}\sqrt{v+\beta_1} \\ \quad = \frac{1}{7}\sin^{-1} C(v^3 + F_1 v^2 + F_2 v + F_3)\sqrt{-v-\beta_2}. \end{cases}$$

and corresponding to $\mu = 18$,

$$(36)\quad \begin{cases} \psi' = \frac{1}{9}\cos^{-1} C(v^4 + E_1 v^3 + E_2 v^2 + E_3 v + E_4)\sqrt{v+\beta_1} \\ \quad = \frac{1}{9}\sin^{-1} C(v^3 + F_1 v^2 + F_2 v + F_3)\sqrt{v^2+1}\sqrt{-v-\beta_2}. \end{cases}$$

21. But if $L^2 - L'^2 = s - \sigma$ is positive we have to use the new variable

$$(1)\quad x = t^2\sqrt{\frac{L+L'}{L-L'}},$$

and make the expression for ψ' in (3), §20, satisfy the differentiation

$$(2)\quad \frac{d\psi'}{dx} = -\frac{x^2 + 2x\dfrac{P}{\sqrt{s-\sigma}} + 1}{2x\sqrt{X_1X_2}},$$

and

$$(3)\quad X_1 = \alpha_1 x^2 + 2\beta_1 x + \gamma_1, \quad X_2 = -\alpha_2 x^2 - 2\beta_2 x - \gamma_2,$$

$$(4)\quad \alpha_1 = \frac{\sqrt{s-s_2} - \sqrt{\sigma - s_2}}{\sqrt{s-\sigma}} = -\gamma_2,$$

$$(5)\quad \alpha_2 = \frac{\sqrt{s-s_2} + \sqrt{\sigma - s_2}}{\sqrt{s-\sigma}} = -\gamma_1,$$

$$(6)\quad \begin{matrix}\beta_1\\ \beta_2\end{matrix} = -\frac{\sqrt{s_1 - s \cdot s - s_3} \pm \sqrt{s_1 - \sigma \cdot \sigma - s_3}}{s-\sigma},$$

$$(7)\quad A_1 = \frac{\sqrt{\alpha_2}}{\lambda}, \quad L_1 = (-1)^r \lambda\sqrt{\alpha_1},$$

$$(8)\quad A_2 = \frac{\sqrt{\alpha_1}}{\lambda}, \quad L_2 = (-1)^r \lambda\sqrt{\alpha_2},$$

but now

$$(9)\quad \lambda^2 = \frac{F\sqrt{s-s_2} + E\sqrt{s_1 - s \cdot s - s_3}}{(s-\sigma)^{n+\frac{1}{2}}} = e^{(2n+1)I(v)},$$

derived from

$$(10)\quad \begin{cases} I(v)=\int \dfrac{P(s-\sigma)+Q}{s-\sigma}\dfrac{ds}{\sqrt{(-S)}} \\ =\dfrac{1}{2n+1}\operatorname{ch}^{-1}\dfrac{F\sqrt{s-s_2}}{(s-\sigma)^{n+\frac{1}{2}}}=\dfrac{1}{2n+1}\operatorname{sh}^{-1}\dfrac{E\sqrt{s_1-s\,.\,s-s_3}}{(s-\sigma)^{n+\frac{1}{2}}}. \end{cases}$$

In this case the degree can be halved by the substitution (*Phil. Trans.*, §24)'

$$(11)\quad q^2=\frac{s-s_2}{s-\sigma},\quad q^2-1=\frac{\sigma-s_2}{s-\sigma},$$

$$(12)\quad s-\sigma=\frac{\sigma-s_2}{q^2-1},\quad s-s_2=(\sigma-s_2)\frac{q^2}{q^2-1},$$

$$(13)\quad s_1-s=\frac{(s_1-\sigma)q^2-(s_1-s_2)}{q^2-1},$$

$$(14)\quad s-s_3=\frac{(\sigma-s_3)q^2-(s_2-s_3)}{q^2-1},$$

$$(15)\quad (s_1-s)(s-s_3)=\frac{(s_1-s_2)(s_2-s_3)}{(q^2-1)^2}Q_1Q_2,$$

$$(16)\quad \begin{matrix}Q_1\\Q_2\end{matrix}=C_2q^2\pm C_1q+1,$$

$$(17)\quad C_2=\sqrt{\frac{s_1-\sigma\,.\,\sigma-s_3}{s_1-s_2\,.\,s_2-s_3}}=\frac{1}{2\sqrt{\alpha(\alpha-m+1)}},$$

$$(18)\quad C_1=\sqrt{\frac{s_1-\sigma}{s_1-s_2}}+\sqrt{\frac{\sigma-s_3}{s_2-s_3}},$$

$$(19)\quad \alpha_1=-\gamma_2=q-\sqrt{q^2-1}=\left(\sqrt{\frac{q+1}{2}}-\sqrt{\frac{q-1}{2}}\right)^2,$$

$$(20)\quad \alpha_2=-\gamma_1=q+\sqrt{q^2-1}=\left(\sqrt{\frac{q+1}{2}}+\sqrt{\frac{q-1}{2}}\right)^2,$$

$$(21)\quad \begin{matrix}\beta_1\\\beta_2\end{matrix}=-\frac{\sqrt{s_1-s_2\,.\,s_2-s_3}\sqrt{Q_1Q_2}\pm\sqrt{s_1-\sigma\,.\,\sigma-s_3}(q^2-1)}{\sigma-s_2},$$

$$(22)\quad \begin{cases} I(v)=\int\left(q^2-1+2P\dfrac{\alpha-m}{m}\right)\dfrac{C_2dq}{\sqrt{(q^2-1)Q_1Q_2}} \\ =\dfrac{1}{n+\frac{1}{2}}\operatorname{ch}^{-1}E_1\sqrt{\dfrac{q+1}{2}}\sqrt{Q_1}=\dfrac{1}{n+\frac{1}{2}}\operatorname{sh}^{-1}E_2\sqrt{\dfrac{q-1}{2}}\sqrt{Q_2} \\ =\dfrac{1}{n+\frac{1}{2}}\log\lambda, \end{cases}$$

$$(23)\quad \lambda=e^{(n+\frac{1}{2})I(v)}=E_1\sqrt{\frac{q+1}{2}}\sqrt{Q_1}+E_2\sqrt{\frac{q-1}{2}}\sqrt{Q_2}.$$

Introducing the appropriate change of sign we have now

(24) $$\frac{B_1}{A_1} = (n - \tfrac{1}{2})\frac{\beta_1}{\alpha_1} + (n + \tfrac{1}{2})\frac{\beta_2}{\alpha_2} - \frac{(2n+1)P}{\sqrt{s - \sigma}},$$

(25) $$\frac{B_2}{A_2} = (n + \tfrac{1}{2})\frac{\beta_1}{\alpha_1} + (n - \tfrac{1}{2})\frac{\beta_2}{\alpha_2} - \frac{(2n+1)P}{\sqrt{s - \sigma}},$$

(26) $$\frac{K_1}{L_1} = (n - \tfrac{1}{2})\frac{\beta_1}{\gamma_1} + (n + \tfrac{1}{2})\frac{\beta_2}{\gamma_2} - \frac{(2n+1)P}{\sqrt{s - \sigma}},$$

(27) $$\frac{K_2}{L_2} = (n + \tfrac{1}{2})\frac{\beta_1}{\gamma_1} + (n - \tfrac{1}{2})\frac{\beta_2}{\gamma_2} - \frac{(2n+1)P}{\sqrt{s - \sigma}},$$

and so on.

Carry s up to s_1, so that

(28) $$v' = \omega_1, \quad v = \omega_1 + \frac{\omega_3}{2n+1},$$

(29) $$\frac{\alpha_1}{\alpha_2} = \frac{\sqrt{s_1 - s_2} \mp \sqrt{\sigma - s_2}}{\sqrt{s_1 - \sigma}} = \frac{-\gamma_2}{-\gamma_1},$$

(30) $$\frac{\beta_1}{\beta_2} = \mp\sqrt{\frac{\sigma - s_3}{s_1 - \sigma}},$$

and we can write

(31) $$X_1 = \quad x^2(\sqrt{s_1 - s_2} - \sqrt{\sigma - s_2}) - 2x\sqrt{\sigma - s_3} - \sqrt{s_1 - s_2} - \sqrt{\sigma - s_3},$$

(32) $$X_2 = -x^2(\sqrt{s_1 - s_2} + \sqrt{\sigma - s_2}) - 2x\sqrt{\sigma - s_3} + \sqrt{s_1 - s_2} - \sqrt{\sigma - s_3}.$$

22. Tested by (*Phil. Trans.*, §29),

(1) $$\mu = 6, \quad v = \omega_1 + \tfrac{1}{3}\omega_3,$$

(2) $$\sigma = 2c, \quad s_2 = 1, \quad P = \tfrac{1}{3}(2c - 1),$$

(3) $$s_1 - s \,.\, s - s_3 = -s^2 + (c^2 + 2c)s - c^2,$$

(4) $$s_1 - \sigma \,.\, \sigma - s_3 = c^2(2c - 1),$$

(5) $$\frac{Q_1}{Q_2} = -cq^2 \pm cq + 1,$$

(6) $$\lambda, \frac{1}{\lambda} = \sqrt{\frac{q+1}{2}}\sqrt{Q_1} \mp \sqrt{\frac{q-1}{2}}\sqrt{Q_2},$$

(7) $$\frac{\beta_1}{\beta_2} = -\frac{\sqrt{Q_1}\sqrt{Q_2} \pm c(q^2 - 1)}{\sqrt{2c - 1}},$$

(8) $$\frac{B_1}{B_2} = -\frac{(q+1)\sqrt{Q_2} \mp (q-1)\sqrt{Q_1}}{\sqrt{2c - 1}},$$

in

$$(9)\quad \begin{cases} \psi' = \tfrac{1}{3}\cos^{-1}(A_1x^2 + B_1x + L_1)\surd X_1 \div Nx^{\frac{3}{2}} \\ \quad = \tfrac{1}{3}\sin^{-1}(A_2x^2 + B_2x + L_2)\surd X_2 \div Nx^{\frac{3}{2}}, \end{cases}$$

$$(10)\quad \frac{d\psi'}{dx} = -\frac{x^2 + \frac{2}{3}x\surd(2c-1)(q^2-1) + 1}{2x\surd X_1X_2},$$

with A, L, α, γ as in (7), (8), (19), (20) §21, and

$$(11)\quad N^2 = \frac{16c^2(q^2-1)^3}{(2c-1)^{\frac{3}{2}}}.$$

Tested by (*Phil. Trans.*, §30),

$$\mu = 10,\quad v = \omega_1 + \tfrac{1}{5}\omega_3,$$

$$(12)\quad P = \tfrac{1}{5}(c+3)(-c^2+4c+1),$$

$$(13)\quad \sigma - s_2 = (c+1)^2(c^2-1)(-c^2+4c+1),$$

$$(14)\quad (s_1-\sigma)(\sigma-s_3) = 16c^2(c+1)^2(c^2-1)(-c^2+4c+1),$$

$$(15)\quad (s_1-s_2)(s_2-s_3) = (c^2-1)^5(-c^2+4c+1),$$

$$(16)\quad \frac{Q_1}{Q_2} = \frac{4cq^2 \pm 4c^2q}{(c+1)(c-1)^2} + 1,$$

$$(17)\quad \lambda, \frac{1}{\lambda} = \left(\frac{2q}{c+1} - 1\right)\sqrt{\frac{q+1}{2}}\surd Q_1 \pm \left(\frac{2q}{c+1} + 1\right)\sqrt{\frac{q-1}{2}}\surd Q_2,$$

$$(18)\quad \frac{\beta_1}{\beta_2} = -\frac{(c+1)(c-1)^2\surd Q_1\surd Q_2 \pm 4c(q^2-1)}{(c+1)\surd(c^2-1)(-c^2+4c+1)}.$$

Putting

$$(19)\quad (q-1)\,[2c(c+3)q^2 - (c^3+3c^2-c+1)q + (c+1)(c^2+1)] = H_1,$$

$$(20)\quad (q+1)\,[2c(c+3)q^2 + (c^3+3c^2-c+1)q + (c+1)(c^2+1)] = H_2,$$

$$(21)\quad \surd(q^2-1)\,[2c(c+3)q^2 - (c^3+c^2-5c-1)q + 2(c+1)] = L_1,$$

$$(22)\quad \surd(q^2-1)\,[2c(c+3)q^2 + (c^3+c^2-5c-1)q + 2(c+1)] = L_2,$$

$$(23)\quad (q+1)\,[4cq^2 - 2c(c-3)q - (c+1)(c-3)] = M_1,$$

$$(24)\quad (q-1)\,[4cq^2 + 2c(c-3)q - (c+1)(c-3)] = M_2,$$

$$(25)\quad \frac{c+1}{c-1}\surd\{(c+1)^3(c-1)(-c^2+4c+1)\} = D,$$

we find

(26) $$DB_1 = (-H_1 - L_1)\sqrt{Q_1} + (+H_2 + L_2)\sqrt{Q_2},$$

(27) $$DB_2 = (+H_1 - L_1)\sqrt{Q_1} + (+H_2 - L_2)\sqrt{Q_2},$$

(28) $$DK_1 = (+H_1 - L_1)\sqrt{Q_1} + (-H_2 + L_2)\sqrt{Q_2},$$

(29) $$DK_2 = (-H_1 - L_1)\sqrt{Q_1} + (-H_2 - L_2)\sqrt{Q_2},$$

(30) $$C_1 = (M_1\sqrt{Q_1} + M_2\sqrt{Q_2})\frac{c-1}{(c+1)(-c^2+4c+1)},$$

(31) $$C_2 = (M_1\sqrt{Q_1} - M_2\sqrt{Q_2})\frac{c-1}{(c+1)(-c^2+4c+1)}$$

(32) $$N^2 = 2048c^4\left(\frac{c-1}{c+1}\right)^2\frac{(q^2-1)^5}{[(c^2-1)(-c^2+4c+1)]^{\frac{5}{2}}}.$$

A change of c into $-1/c$ will give a parameter

$$v = \omega_1 + \tfrac{3}{5}\omega_3 \text{ in the region } \sqrt{5}+2 > c > 1,$$

with

(33) $$P = \tfrac{1}{5}(3c-1)(-c^2+4c+1),$$

(34) $$\sigma - s_2 = (c-1)^2(c^2-1)(-c^2+4c+1),$$

(35) $$s_1 - \sigma \,.\, \sigma - s_3 = 16c^4(c-1)^2(c^2-1)(-c^2+4c+1),$$

(36) $$\lambda, \frac{1}{\lambda} = \pm\left(\frac{2cq}{c-1} - 1\right)\sqrt{\frac{q+1}{2}}\sqrt{Q_1} + \left(\frac{2cq}{c-1} + 1\right)\sqrt{\frac{q-1}{2}}\sqrt{Q_2},$$

(37) $$\begin{matrix}Q_1\\Q_2\end{matrix} = \frac{4c^2q^2 \mp 4cq}{(c-1)(c+1)^2} - 1,$$

(38) $$\begin{matrix}\beta_1\\\beta_2\end{matrix} = -\frac{(c-1)(c+1)^2\sqrt{Q_1}\sqrt{Q_2} \mp 4c^2(q^2-1)}{(c-1)\sqrt{(c^2-1)(-c^2+4c+1)}}, \text{ \&c.}$$

We can construct the special case in which

(39) $$v = \omega_1 + \tfrac{1}{5}\omega_3, \quad v' = \omega_1 + \tfrac{3}{5}\omega_3,$$

by taking

(40) $$L'^2 - L^2 = \sigma - s = 4c(c^2-1)(-c^2+4c+1),$$

and now

(41) $$\frac{q^2}{q^2-1} = \left(\frac{c+1}{c-1}\right)^2, \quad q = \frac{c+1}{2\sqrt{c}},$$

(42) $$\frac{q+1}{2}Q_1 = \frac{q-1}{2}Q_2 = \frac{1}{4\sqrt{c}}\cdot\frac{c-1}{c+1},$$

(43) $$\lambda = \sqrt[4]{c}\sqrt{\frac{c+1}{c-1}},$$

(44) $$\alpha_1 = \frac{1}{\sqrt{c}} = -\gamma_2, \quad \alpha_2 = \sqrt{c} = -\gamma_1,$$

(45) $$\beta_1 = \frac{c^2+1}{\sqrt{(c^2-1)(-c^2+4c+1)}}, \quad \beta_2 = \frac{-c^2+2c+1}{\sqrt{(c^2-1)(-c^2+4c+1)}},$$

and we find

$$(46)\begin{cases} \psi' = \tfrac{1}{5}\cos^{-1}\Big[x^4\sqrt{\dfrac{c-1}{c+1}} + x^3\dfrac{(c-1)^2(2c+1)}{(c+1)\sqrt{c}\sqrt{-c^2+4c+1}} \\ \quad + x^2\dfrac{-(c-1)(c-2)}{\sqrt{c^2-1}} + x\dfrac{2\sqrt{c}(c-2)}{\sqrt{-c^2+4c+1}} + \sqrt{\dfrac{c+1}{c-1}}\Big]\sqrt{X_1} \div Nx^{\frac{5}{2}}, \\ = \tfrac{1}{5}\sin^{-1}\Big[\dfrac{x^4}{\sqrt{c}}\sqrt{\dfrac{c-1}{c+1}} + x^3\dfrac{(c-1)(3c+1)}{(c+1)\sqrt{-c^2+4c+1}} \\ \quad + x^2\dfrac{\sqrt{c}(c-1)(3c^2-8c+1)}{(-c^2+4c+1)\sqrt{c^2-1}} - x\dfrac{c^2-4c+1}{\sqrt{-c^2+4c+1}} - \sqrt{c}\sqrt{\dfrac{c+1}{c-1}}\Big]\sqrt{X_2} \div Nx^{\frac{5}{2}}, \end{cases}$$

$$(47)\qquad N^2 = \frac{2(c+1)^6}{c(c^2-1)^{\frac{1}{2}}(-c^2+4c+1)^{\frac{5}{2}}},$$

satisfying the differentiation (15), §20; at the same time ϕ' is also of a similar form, so that α_1, β_1, γ_1, δ_1, are quasi-algebraical.

A change of x into xi will give the case in which

$$(48)\qquad L^2 - L'^2 = s - \sigma = 4c(c^2-1)(c^2-4c-1),$$

is positive, as in the region

$$-\sqrt{5} + 2 > c > -1,$$

where

$$(49)\qquad v = \omega_1 + \tfrac{4}{5}\omega_3, \quad v' = \omega_1 + \tfrac{2}{5}\omega_3;$$

and now we take

$$(50)\ X_1 = \frac{x^2}{\sqrt{-c}} - 2x\frac{c^2+0+1}{\sqrt{(-c^2+1)(c^2-4c-1)}} - \sqrt{-c},$$

$$(51)\ X_2 = -x^2\sqrt{-c} + 2x\frac{c^2-2c-1}{\sqrt{(-c^2+1)(c^2-4c-1)}} + \frac{1}{\sqrt{-c}},$$

$$(52)\begin{cases} \psi' = \tfrac{1}{5}\cos^{-1}\Big[x^4\sqrt{\dfrac{-c+1}{c+1}} + x^3\dfrac{(-c+1)^2(2c+1)}{(c+1)\sqrt{-c}\sqrt{c^2-4c-1}} \\ \quad + x^2(c-2)\sqrt{\dfrac{-c+1}{c+1}} + x\dfrac{2\sqrt{-c}(c-2)}{\sqrt{c^2-4c-1}} - \sqrt{\dfrac{c+1}{-c+1}}\Big]\sqrt{X_1} \div Nx^{\frac{5}{2}} \\ = \tfrac{1}{5}\sin^{-1}\Big[\dfrac{x^4}{\sqrt{-c}}\sqrt{\dfrac{-c+1}{c+1}} + x^3\dfrac{(-c+1)(3c+1)}{(c+1)\sqrt{c^2-4c-1}} \\ \quad - x^2\sqrt{-c}\dfrac{3c^2-8c+1}{c^2-4c-1}\sqrt{\dfrac{-c+1}{c+1}} - x\dfrac{c^2-4c+1}{\sqrt{c^2-4c-1}} - \sqrt{-c}\sqrt{\dfrac{c+1}{-c+1}}\Big]\sqrt{X_2} \div Nx^{\frac{5}{2}}, \end{cases}$$

$$(53)\qquad N^2 = \frac{2(c+1)^6}{-c(-c^2+1)^{\frac{1}{2}}(c^2-4c-1)^{\frac{5}{2}}},$$

satisfying, as in (2) §21,

(54)
$$\frac{d\psi'}{dx} = -\frac{x^2 + 2x\dfrac{(c+3)(c^2-4c-1)}{5\sqrt{-c}\sqrt{(-c^2-1)(c^2-4c-1)}} + 1}{2x\sqrt{X_1X_2}}.$$

According to *Phil. Trans.*, §31,

(55)
$$\mu = 14, \quad v = \omega_1 + \tfrac{1}{7}\omega_3,$$

we have

(56)
$$\begin{cases} \lambda = e^{\frac{7}{2}I(v)} \\ = (A_2q^2 + A_1q + 1)\sqrt{\dfrac{q+1}{2}}\sqrt{Q_1} + (A_2q^2 - A_1q + 1)\sqrt{\dfrac{q-1}{2}}\sqrt{Q_2}, \end{cases}$$

(57)
$$\begin{matrix} Q_1 \\ Q_2 \end{matrix} = C_2q^2 \pm C_1q + 1,$$

and putting $N_7 = 0$ in *Phil. Trans.* (16), p. 248,

(58)
$$\alpha = \frac{m(1-m)(2-3m+\sqrt{M})}{2(1-2m)^2}, \quad M = 4m - 11m^2 + 8m^3;$$

thence

(59)
$$C_2 = \frac{(1-2m)(-m+\sqrt{M})}{4m(1-m)^3},$$

(60)
$$C_1 = -\frac{(1-2m)(3m-4m^2+\sqrt{M})}{4m(1-m)^3},$$

(61)
$$A_2 = \frac{-2+3m+\sqrt{M}}{2m(1-m)^2},$$

(62)
$$A_1 = \frac{m-2m^2-\sqrt{M}}{2m(1-m)^2},$$

(63)
$$7P(v) = \frac{3m+2m^2-\sqrt{M}}{2m};$$

and here is material enough to build up the corresponding case of

(64)
$$\begin{cases} \psi' = \tfrac{1}{7}\cos^{-1}(A_1x^6 + B_1x^5 + \dots + K_1x + L_1)\sqrt{X_1} \div Nx^{\frac{7}{2}} \\ = \tfrac{1}{7}\sin^{-1}(A_2x^6 + B_2x^5 + \dots + K_2x + L_2)\sqrt{X_2} \div Nx^{\frac{7}{2}}; \end{cases}$$

but A_1, A_2 and C_1, C_2 are used here in a meaning different to that above.

Similarly, from *Phil. Trans.*, §32, or *Archiv der Mathematik und Physik*, III, Serie I, p. 73, for

(65)
$$\mu = 18, \quad v = \omega_1 + \tfrac{1}{9}\omega_3,$$

the expression for λ can be written down in the form

$$(66)\left\{\begin{aligned}\lambda &= e^{\frac{3}{2}I}\\ &=(A_3q^3+A_2q^2+A_1q+1)\sqrt{\frac{q+1}{2}}\sqrt{Q_1}+(A_3q^3-A_2q^2+A_1q-1)\sqrt{\frac{q-1}{2}}\sqrt{Q_2},\end{aligned}\right.$$

$$(67)\ \begin{matrix}Q_1\\Q_2\end{matrix} = C_2q^2 \pm C_1q + 1,$$

and the values of A_1, A_2, A_3, and C_1, C_2, are given in *Phil. Trans.*, p. 271; and thence the construction of the case

$$(68)\quad\left\{\begin{aligned}\psi' &= \tfrac{1}{9}\cos^{-1}(A_1x^8 + B_1x^7 + \ldots\ldots + K_1x + L_1)\sqrt{X_1} \div Nx^{\frac{9}{2}}\\ &= \tfrac{1}{9}\sin^{-1}(A_2x^8 + B_2x^7 + \ldots\ldots + K_2x + L_2)\sqrt{X_2} \div Nx^{\frac{9}{2}},\end{aligned}\right.$$

in which the two meanings of A_1, A_2, and C_1, C_2, must be kept distinct.

So also theoretically for $\mu = 22, \ldots\ldots$

23. For the determination of the motion of the centre O with μ even, we proceed as before for odd μ, and the expression for ψ' in terms of $t = \tan\frac{1}{2}\vartheta$ will serve for ϖ', if L' is taken to mean L'', when $k = \lambda^2$ is positive.

Replacing this t by $\dfrac{w+1}{-w+1}$, and putting

$$(1)\quad\left\{\begin{aligned}W_1 &= (w-1)^4T_1\\ &= \alpha_1(w+1)^4 + 2\beta_1(w^2-1)^2 + \gamma_1(w-1)^4\\ &= (\alpha_1 + 2\beta_1 + \gamma_1)(w^4+1)\\ &\quad + 4(\alpha_1 - \gamma_1)(w^3 + w)\\ &\quad + (6\alpha_1 - 4\beta_1 + 6\gamma_1)w^2,\end{aligned}\right.$$

in which we can take, as a typical case,

$$(2)\qquad \alpha_1 = (L+L')\frac{\sqrt{s-s_2} - \sqrt{\sigma - s_2}}{\sqrt{s-\sigma}},$$

$$(3)\qquad \gamma_1 = -(L-L')\frac{\sqrt{s-s_2} + \sqrt{\sigma-s_2}}{\sqrt{s-\sigma}},$$

$$(4)\qquad \beta_1 = -\frac{\sqrt{s_1-\sigma\,.\,\sigma-s_3} + \sqrt{s_1-s\,.\,s-s_3}}{\sqrt{s-\sigma}},$$

so that

$$(5)\qquad \alpha_1 - \gamma_1 = 2\frac{L\sqrt{s-s_2} - L'\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}},$$

$$(6)\qquad \alpha_1 + \gamma_1 = 2\frac{L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}},$$

$$(7)\quad \alpha_1 + 2\beta_1 + \gamma_1 = 2\frac{L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2} - \sqrt{s_1-\sigma\,.\,\sigma-s_3} - \sqrt{s_1-s\,.\,s-s_3}}{\sqrt{s-\sigma}},$$

$$(8)\ 6\alpha_1 - 4\beta_1 + 6\gamma_1 = 4\frac{3(L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2}) + \sqrt{s_1-\sigma\,.\,\sigma-s_3} + \sqrt{s_1-s\,.\,s-s_3}}{\sqrt{s-\sigma}},$$

and so also for W_2; and then

$$(9) \qquad \frac{d\varpi'}{dw} = -2\frac{(L+P)(-w^2+1)^2+4L'(w^3+w)+8Lw^2}{(-w^2+1)\sqrt{(W_1W_2)}}$$

But for negative k we replace w, L', and $\sqrt{s-s_2}$ by wi, $-L'i$, and $\sqrt{s_2-s}$; and obtain a real form for ϖ', leading to the differential relation

$$(10) \qquad \frac{d\varpi'}{dw} = -2\frac{(L+P)(w^2+1)^2-2L'(w^3-w)-8Lw^2}{(w^2+1)\sqrt{(W_1W_2)}},$$

24. Returning to the original variable z, and considering the resolution of the quartic

$$(1) \qquad \begin{cases} Z = az^4+4bz^3+6cz^2+4dz+e \\ \quad = a(z^2-1)\left(z^2-4\frac{B}{M}z-1-aD\right)-4\left(\frac{L'z-L}{M}\right)^2 \\ \quad = a(z^2-1)\left(z^2-4\frac{B}{M}z-1-aE\right)-4\left(\frac{Lz-L'}{M}\right)^2, \end{cases}$$

denote the roots by δ, α, β, γ, so that

$$(2) \qquad \delta+\alpha+\beta+\gamma = 4\frac{B}{M},$$

$$(3) \qquad (\delta+\alpha)(\beta+\gamma)+\delta\alpha+\beta\gamma = -2-aD-4a\frac{L'^2}{M^2} = -2-aE-4a\frac{L^2}{M^2},$$

$$(4) \qquad \delta(\beta\gamma+\gamma\alpha+\alpha\beta)+\alpha\beta\gamma = -4\frac{B}{M}-8a\frac{LL'}{M^2},$$

$$(5) \qquad \delta\alpha\beta\gamma = 1+aD-4a\frac{L^2}{M^2} = 1+aE-4a\frac{L'^2}{M^2}.$$

Then if x, y, denote any two values of z corresponding to the values u_1, u_2, of the elliptic argument

$$(6) \qquad u = \int\frac{dz}{\sqrt{Z}}.$$

Laguerre's formula gives

$$(7) \quad \wp(u_1\pm u_2)-e_\alpha = \left[\frac{\sqrt{\alpha.x-\delta.x-\alpha.y-\beta.y-\gamma)}\mp\sqrt{(\alpha.y-\delta.y-\alpha.x-\beta.x-\gamma)}}{2(x-y)}\right].$$

Thus v_1 and v_2 denoting the values of u corresponding to $z = +1$ and $z = -1$,

$$(8)\quad \begin{cases} \wp(v_1 \pm v_2) - e_\alpha \\ \quad = \frac{1}{8}a[1 - (\delta + \alpha)(\beta + \gamma) + \delta\alpha + \beta\gamma + \delta\alpha\beta\gamma] \mp \dfrac{L^2 - L'^2}{2M^2} \\ \quad = \frac{1}{8}a\left[1 - 2(\delta + \alpha)(\beta + \gamma) - 2 - aE - 4a\dfrac{L^2}{M^2} + 1 + aE - 4a\dfrac{L'^2}{M^2}\right] \\ \qquad\qquad \mp \dfrac{L^2 - L'^2}{2M^2} \\ \quad = -\frac{1}{4}a(\delta + \alpha)(\beta + \gamma) - \dfrac{L^2 \text{ or } L'^2}{M^2}; \end{cases}$$

and denoting $v_1 + v_2$ by v and $v_1 - v_2$ by v', as before,

$$(9)\qquad \wp v - e_\alpha = -\tfrac{1}{4}a(\delta + \alpha)(\beta + \gamma) - \frac{L^2}{M^2},$$

$$(10)\qquad \wp v' - e_\alpha = -\tfrac{1}{4}a(\delta + \alpha)(\beta + \gamma) - \frac{L'^2}{M^2}.$$

The factor M is at our disposal as a *homogeneity factor*, so put

$$(11)\qquad \wp v - e_\alpha = \frac{\sigma - s_\alpha}{M^2},$$

in accordance with the notation employed in *Phil. Trans.*, 1904; then

$$(12)\qquad \tfrac{1}{4}M^2(\delta + \alpha)(\beta + \gamma) = -aL^2 - a(\sigma - s_\alpha),$$

and

$$(13)\qquad \tfrac{1}{16}M^2[(\delta + \alpha) + (\beta + \gamma)]^2 = B^2,$$

so that

$$(14)\qquad \tfrac{1}{16}M^2[(\delta + \alpha) - (\beta + \gamma)]^2 = B^2 + aL^2 + a(\sigma - s_\alpha) = N_\alpha^2 \text{ suppose}$$

$$(15)\qquad \tfrac{1}{4}M[(\delta + \alpha) - (\beta + \gamma)] = N_\alpha,$$

$$(16)\qquad \tfrac{1}{2}M(\delta + \alpha) = B - N_\alpha,$$

$$(17)\qquad \tfrac{1}{2}M(\beta + \gamma) = B + N_\alpha,$$

$$(18)\qquad M\delta = B - N_\alpha - N_\beta - N_\gamma,$$

$$(19)\qquad M\alpha = B - N_\alpha + N_\beta + N_\gamma,$$

$$(20)\qquad M\beta = B + N_\alpha - N_\beta + N_\gamma,$$

$$(21)\qquad M\gamma = B + N_\alpha + N_\beta - N_\gamma.$$

Then

$$(22)\qquad \wp v - e_\alpha = -\frac{aB^2 + L^2}{M^2} + \frac{aN_\alpha^2}{M^2},$$

and since from (19), §7,

$$(23)\qquad \wp 2w - \wp v = \frac{aB^2 + L^2}{M^2},$$

therefore

$$(24)\qquad \wp 2w - e_\alpha = \frac{aN_\alpha^2}{M^2},$$

w denoting the value of u corresponding to $z = \infty$.

Thus $v = 2w$ if $aB^2 + L^2 = 0$, implying $a = -1$.

Denoting $z - \delta \,.\, z - \alpha$ by Z_1 and $z - \beta \,.\, z - \gamma$ by Z_2,

$$(25)\qquad Z_1 = \left(z - \frac{B}{M}\right)^2 - 2\frac{N_\alpha}{M}\left(z - \frac{B}{M}\right) + \frac{N_\alpha^2 - (N_\beta + N_\gamma)^2}{M^2},$$

$$(26)\qquad Z_2 = \left(z - \frac{B}{M}\right)^2 + 2\frac{N_\alpha}{M}\left(z - \frac{B}{M}\right) + \frac{N_\alpha^2 - (N_\beta - N_\gamma)^2}{M^2},$$

and

$$(27)\qquad \left\{\begin{aligned} Z &= aZ_1Z_2 \\ &= a\left[\left(z - \frac{B}{M}\right)^4 - 2\frac{N_1^2 + N_2^2 + N_3^2}{M^2}\left(z - \frac{B}{M}\right)^2 - 8\frac{N_1N_2N_3}{M^3}\left(z - \frac{B}{M}\right)\right. \\ &\quad \left. + \frac{N_1^4 + N_2^4 + N_3^4 - 2N_2^2N_3^2 - 2N_3^2N_1^2 - 2N_1^2N_2^2}{M^4}\right], \end{aligned}\right.$$

and, therefore, equating the coefficients of the powers of z,

$$(28)\qquad 6a\frac{B^2}{M^2} - 2a\frac{N_1^2 + N_2^2 + N_3^2}{M^2} = 6c = -2a - E - 4\frac{L^2}{M^2},$$

so that

$$(29)\qquad 2a\frac{N_1^2 + N_2^2 + N_3^2}{M^2} = 2a + E + 4\frac{L^2}{M^2} + 6a\frac{B^2}{M^2} = 6\wp 2w,$$

a verification;

$$(30)\qquad -4a\frac{B^3}{M^3} + 4a\frac{N_1^2 + N_2^2 + N_3^2}{M^2}\frac{B}{M} - 8a\frac{N_1N_2N_3}{M^3} = 4d = 4a\frac{B}{M} + 8\frac{LL'}{M^2},$$

$$(31)\qquad a\frac{B^4}{M^4} - 2\frac{N_1^2 + N_2^2 + N_3^2}{M^2}\frac{B^2}{M^2} + 8\frac{N_1N_2N_3}{M^3}\frac{B}{M} + \frac{N_1^4 + \ldots\ldots - 2N_2^2N_3^2\ldots\ldots}{M^4}$$

$$= e = a + E - 4\frac{L'^2}{M^2}.$$

From (10), §7,

$$(32)\qquad 2aBLL'M = iLM^3\wp'v - \tfrac{1}{2}EL^2M^2 = aL\surd - \Sigma - \tfrac{1}{2}EL^2M^2,$$

so that from (28),

$$(33)\qquad 2aBLL'M = aL\surd - \Sigma - L^2[a(N_1^2 + N_2^2 + N_3^2) - aM^2 - 2L^2 - 3aB^2],$$

and from (30)

$$(34)\qquad 2aBLL'M = -B^2M^2 - B^4 + (N_1^2 + N_2^2 + N_3^2)B^2 - 2N_1N_2N_3B,$$

so that

$$(35)\qquad \left\{\begin{aligned} (B^2 + aL^2)M^2 = (B^2 + aL^2)(N_1^2 + N_2^2 + N_3^2) - B^4 - 3aB^2L^2 - 2L^4 \\ - 2B_1N_1N_2N_3 - aL\surd - \Sigma, \end{aligned}\right.$$

$$(36)\qquad \left\{\begin{aligned} M^2 &= N_1^2 + N_2^2 + N_3^2 - B^2 - 2aL^2 \frac{-2BN_1N_2N_3 + aL\surd - \Sigma}{B^2 + aL^2} \\ &= N_1^2 + N_2^2 + N_3^2 - B^2 - 2aL^2 + \frac{2aBN_1N_2N_3 - L\surd - \Sigma}{aB^2 + L^2}; \end{aligned}\right.$$

and we should obtain the same value of M^2 by equating the coefficients of z^0 in (27).

This determination of the homogeneity factor M^2 is essential in the applications, when we build up a solution on arbitrary values assigned to B, L, and σ, s_1, s_2, s_3, as will be seen in the sequel.

The expression for M^2 appears rather complicated and inelegant, but not much more than the form required in the corresponding case of the motion of a top (*Annals of Mathematics*, vol. 5, 1904).

The expression for M^2 can also be thrown into the form

$$(37)\qquad M^2 = \frac{(BN_\alpha - N_\beta N_\gamma)^2 - (L\sqrt{-\sigma_\alpha} + \sqrt{\sigma_\beta\sigma_\gamma})^2}{B^2 + aL^2 = N_\alpha^2 - a\sigma_\alpha} = a[s(2w) - s(v)].$$

Putting

$$(38)\qquad \wp(2w) - \wp(v) = s(2w) - s(v) = s' - \sigma,$$

and choosing $s_\alpha = s_2$, we can now write

$$(39)\qquad aM^2 = \frac{(B\sqrt{s' - s_2} - \sqrt{s_1 - s' . s' - s_3})^2 + (L\sqrt{s_2 - \sigma} - \sqrt{s_1 - \sigma . \sigma - s_3})^2}{s' - \sigma}.$$

25. Going back again to the variable t, of §12, and working with the case of $s_a = s_2$, and

$$(1)\quad \begin{cases} T_1 = (L+L')\dfrac{\sqrt{s-s_2}-\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}}\,t^4 \\ \qquad -2\dfrac{\sqrt{s_1-s\,.\,s-s_3}+\sqrt{s_1-\sigma\,.\,\sigma-s_3}}{\sqrt{s-\sigma}}\,t^2 \\ \qquad -(L-L')\dfrac{\sqrt{s-s_2}+\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}} \end{cases}$$

$$(2)\quad \begin{cases} T_2 = -(L+L')\dfrac{\sqrt{s-s_2}+\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}}\,t^4 \\ \qquad +2\dfrac{\sqrt{s_1-s\,.\,s-s_3}-\sqrt{s_1-\sigma\,.\,\sigma-s_3}}{\sqrt{s-\sigma}}\,t^2 \\ \qquad +(L-L')\dfrac{\sqrt{s-s_2}-\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}}, \end{cases}$$

then put

$$(3)\qquad t^2 = \frac{1-z}{1+z},\quad \frac{t^2}{t^2+1} = \frac{1-z}{2},\quad \frac{1}{t^2+1} = \frac{1+z}{2},$$

$$(4)\qquad M^2Z = aM^2Z_1Z_2 = \frac{4T_1T_2}{(t^2+1)^4},$$

and identify with

$$(5)\quad \begin{cases} M_1Z_1 = \dfrac{2T_1}{(t^2+1)^2} \\ \quad = (L+L')\dfrac{\sqrt{s-s_2}-\sqrt{\sigma-s_2}}{2\sqrt{s-\sigma}}(z-1)^2 \\ \qquad + \dfrac{\sqrt{s_1-s\,.\,s-s_3}+\sqrt{s_1-\sigma\,.\,\sigma-s_3}}{\sqrt{s-\sigma}}(z^2-1) \\ \quad -(L-L')\dfrac{\sqrt{s-s_2}+\sqrt{\sigma-s_2}}{2\sqrt{s-\sigma}}(z+1)^2, \end{cases}$$

and

$$(6)\qquad Z_1 = \left(z-\frac{B}{M}\right)^2 - 2\frac{N_2}{M}\left(z-\frac{B}{M}\right) + \frac{N_2^2-(N_1+N_3)^2}{M^2},$$

so that, equating coefficients of z^2, $-2z$, and z^0,

(7) $$M_1 = \frac{L'\sqrt{s-s_2} - L\sqrt{\sigma - s_2} + \sqrt{s_1 - s \,.\, s - s_3} + \sqrt{s_1 - \sigma \,.\, \sigma - s_3}}{\sqrt{s-\sigma}},$$

(8) $$\frac{B+N_2}{M} = \frac{L\sqrt{s-s_2} - L'\sqrt{\sigma - s_2}}{M_1\sqrt{s-\sigma}},$$

(9) $$\left(\frac{B+N_2}{M}\right)^2 - \left(\frac{N_1+N_3}{M}\right)^2 = \frac{L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2} - \sqrt{s_1-s \,.\, s-s_3} - \sqrt{s_1-\sigma \,.\, \sigma-s_3}}{M_1\sqrt{s-\sigma}}.$$

Similarly, with

(10) $$M_2 Z_2 = \frac{2T_2}{(t^2+1)^2},$$

(11) $$Z_2 = \left(z - \frac{B}{M}\right)^2 + 2\frac{N_2}{M}\left(z - \frac{B}{M}\right) + \frac{N_2^2 - (N_1 - N_3)^2}{M^2},$$

(12) $$M_2 = \frac{-L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2} - \sqrt{s_1-s \,.\, s-s_3} + \sqrt{s_1-\sigma \,.\, \sigma-s_3}}{\sqrt{s-\sigma}},$$

(13) $$\frac{B-N_2}{M} = \frac{-L\sqrt{s-s_2} - L'\sqrt{\sigma-s_2}}{M_2\sqrt{s-\sigma}},$$

(14) $$\left(\frac{B-N_2}{M}\right)^2 - \left(\frac{N_1-N_3}{M}\right)^2 = \frac{-L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2} + \sqrt{s_1-s \,.\, s-s_3} - \sqrt{s_1-\sigma \,.\, \sigma-s_3}}{M_2\sqrt{s-\sigma}}.$$

Then

(15) $$M_1 + M_2 = 2\frac{-L\sqrt{\sigma-s_2} + \sqrt{s_1-\sigma \,.\, \sigma-s_3}}{\sqrt{s-\sigma}},$$

(16) $$M_1 - M_2 = 2\frac{L'\sqrt{s-s_2} + \sqrt{s_1-s \,.\, s-s_3}}{\sqrt{s-\sigma}},$$

(17) $$\left\{\begin{aligned} aM^2 = M_1 M_2 &= \frac{(L\sqrt{\sigma-s_2} - \sqrt{s_1-\sigma \,.\, \sigma-s_3})^2 - (L'\sqrt{s-s_2} + \sqrt{s_1-s \,.\, s-s_3})^2}{s-\sigma} \\ &= -L^2 + 2(s-\sigma) + 3M^2\wp v - \frac{L'\sqrt{-S} + Lxy}{s-\sigma}, \end{aligned}\right.$$

analogous to the form in (36), §24.

Also

(18) $$\left\{\begin{aligned} aM(B+N_2) = {} & \frac{L\sqrt{s-s_2} - L'\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}} \\ & \times \frac{-L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2} - \sqrt{s_1-s \,.\, s-s_3} + \sqrt{s_1-\sigma \,.\, \sigma-s_3}}{\sqrt{s-\sigma}}, \end{aligned}\right.$$

(19) $$\left\{\begin{aligned} aM(B-N_2) = {} & \frac{-L\sqrt{s-s_2} - L'\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}} \\ & \times \frac{L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2} + \sqrt{s_1-s \,.\, s-s_3} + \sqrt{s_1-\sigma \,.\, \sigma-s_3}}{\sqrt{s-\sigma}}, \end{aligned}\right.$$

and adding,

(20) $$2aBM = -2LL' - \frac{L\sqrt{-S} + L'\sqrt{-\Sigma}}{s-\sigma}.$$

Subtracting

(21) $$\left\{\begin{aligned} 2MN_2 &= 2\sqrt{s-s_2 \,.\, \sigma-s_2} \\ &\quad + 2\frac{-L\sqrt{s-s_2}\sqrt{s_1-\sigma \,.\, \sigma-s_3} - L'\sqrt{\sigma-s_2}\sqrt{s_1-s \,.\, s-s_3}}{s-\sigma}. \end{aligned}\right.$$

Again

(22) $$a(B+N_2)^2 - a(N_1+N_3)^2 = \frac{(L\sqrt{\sigma-s_2}+\sqrt{s_1-s \,.\, s-s_3})^2 - (L'\sqrt{s-s_2}-\sqrt{s_1-\sigma \,.\, \sigma-s_3})^2}{s-\sigma},$$

(23) $$a(B-N_2)^2 - a(N_1-N_3)^2 = \frac{(L\sqrt{\sigma-s_2}-\sqrt{s_1-s \,.\, s-s_3})^2 - (L'\sqrt{s-s_2}+\sqrt{s_1-\sigma \,.\, \sigma-s_3})^2}{s-\sigma},$$

and subtracting

(24) $$a(BN_2 - N_1N_3) = \frac{L\sqrt{\sigma-s_2}\sqrt{s_1-s \,.\, s-s_3} + L'\sqrt{s-s_2}\sqrt{s_1-\sigma \,.\, \sigma-s_3}}{s-\sigma},$$

while adding leads to an identity.

From the two equations (24) and

(25) $$2(aBM + LL') = -\frac{L\sqrt{-S} + L'\sqrt{-\Sigma}}{s-\sigma},$$

squaring and subtracting

(26) $$\left\{\begin{aligned} (aBM + LL')^2 - (BN_2 - N_1N_3)^2 &= \frac{L^2(s_1-s \,.\, s-s_3) - L'^2(s_1-\sigma \,.\, \sigma-s_3)}{s-\sigma}, \\ &= L^2(s_1 - s + \sigma - s_3) + (s_1-\sigma)(\sigma-s_3), \end{aligned}\right.$$

(27) $$\left\{\begin{aligned} 2aBLL'M &= -B^2M^2 - L^2L'^2 + (BN_2 - N_1N_3)^2 \\ &\quad + L^2(s_1 - s + \sigma - s_3) + (s_1-\sigma)(\sigma-s_3) \end{aligned}\right.$$

and eliminating $\sqrt{-S}$ between (17) and (25)

(28) $$2aBLL'M = -aL^2M^2 - L^4 - L^2(3\sigma - s_1 - s_2 - s_3) - L\sqrt{-\Sigma},$$

so that, eliminating $2aBLL'M$,

(29) $$(B^2 + aL^2)M^2 = (BN_2 - N_1N_3)^2 + (L\sqrt{\sigma-s_2} - \sqrt{s_1-\sigma \,.\, \sigma-s_3})^2,$$

as before in (37), §24.

26. With $aB^2 + L^2 = 0$, implying $a = -1$, equation (20), §7, shows that

(1) $$\wp 2w - \wp v = 0, \quad 2w = v = v_1 + v_2,$$

and now, with

(2) $$\begin{cases} Z = -z^4 + 4bz^3 + 6cz^2 + 4dz + e. \\ \quad = -(z^2-1)\left(z^2 - 4\dfrac{B}{M}z - 1 + E\right) - 4\left(\dfrac{Lz - L'}{M}\right)^2, \end{cases}$$

(3) $$b = \frac{B}{M}, \quad b + d = 2\frac{LL'}{M^2},$$

(4) $$1 + \frac{d}{b} = \pm\frac{2L'}{M}, \quad \text{as } B = \pm L,$$

(5) $$6c = 2 - E - 4b^2,$$

(6) $$e = -1 + E - \left(1 + \frac{d}{b}\right)^2,$$

(7) $$6c + e = 1 - 4b^2 - \left(1 + \frac{d}{b}\right)^2,$$

(8) $$\wp(v_1 - v_2) = -c - \tfrac{1}{4}\left(1 + \frac{d}{b}\right)^2,$$

(9) $$\wp(v_1 + v_2) = -c - b^2,$$

(10) $$i\wp'(v_1 - v_2) = \frac{(e+1)(b+d) - 4b^2(b-d)}{\pm 4b}, \quad i\wp'(v_1 + v_2) = \frac{b^2e + d^2}{\pm 2b},$$

whence

(11) $$\wp\tfrac{1}{2}(v_1 - v_2) = \tfrac{1}{2}\left(c + \frac{d}{b}\right),$$

(12) $$\wp\tfrac{1}{2}(v_1 + v_2) = \wp w = \tfrac{1}{2}(ad^2 + 2bd + c);$$

and

(13) $$\wp(u - w) = \frac{-z^2 + 2bz + c + \sqrt{(-Z)}}{2},$$

(14) $$\wp\left(u - \frac{v_1 + v_2}{2}\right) - \wp\frac{v_1 - v_2}{2} = \frac{-z^2 + 2bz - \dfrac{d}{b} + \sqrt{(-Z)}}{2}.$$

Now if we put

$$(15)\quad \begin{cases} \chi = \tan^{-1} \dfrac{\sqrt{Z}}{-z^2 + 2bz - \dfrac{d}{b}} \\[2ex] \quad = \sin^{-1} \dfrac{\sqrt{Z}}{\sqrt{\left(\dfrac{d^2}{b^2} + e\right)}\sqrt{(-z^2+1)}} = \cos^{-1} \dfrac{-z^2 + 2bz - \dfrac{d}{b}}{\sqrt{\left(\dfrac{d^2}{b^2}+e\right)}\sqrt{(-z^2+1)}}, \end{cases}$$

we find on differentiation

$$(16)\qquad \frac{d\chi}{dz} = \frac{z}{\sqrt{Z}} + \frac{\left(1 + \dfrac{d}{b}\right)z - 2b}{(-z^2+1)\sqrt{Z}} = \frac{z}{\sqrt{Z}} \mp \frac{d(\psi - qFt)}{dz},$$

$$(17)\qquad \chi \pm (\psi - qFt) = \int \frac{zdz}{\sqrt{Z}},$$

according as $B = \pm L$; and

$$(18)\qquad i\left(z - \frac{B}{M}\right) = \zeta(v+w) - \zeta(v-w) - \zeta 2w,$$

so that

$$(19)\qquad \int \frac{zdz}{\sqrt{Z}} = \frac{B}{M} nt + i \log \frac{\sigma(v-w)}{\sigma(v+w)} e^{v\zeta 2w},$$

and algebraical cases can be constructed by choosing w an aliquot part of a period.

Now

$$(20)\qquad N_a^2 = s_a - \sigma = -\sigma_a \text{ suppose, and } 2N_1N_2N_3 = \sqrt{-\Sigma},$$

$$(21)\qquad \frac{2BN_1N_2N_3 - L\sqrt{-\Sigma}}{B^2 - L^2} = \frac{0}{0} = \frac{2B^2\wp''v - \wp'^2v}{-2Bi\wp'v},$$

$$(22)\qquad M^2 = B^2 + N_1^2 + N_2^2 + N_3^2 + \frac{B^2\wp''v - \frac{1}{2}\wp'^2v}{Bi\wp'v},$$

$$(23)\quad \begin{cases} BM^2 = B^3 + B^2\dfrac{\wp''v}{i\wp'v} - 3B\wp v + \frac{1}{2}i\wp'v \\[2ex] \quad = \left(B + \sqrt{-\dfrac{\sigma_2\sigma_3}{\sigma_1}}\right)\left(B + \sqrt{-\dfrac{\sigma_3\sigma_1}{\sigma_2}}\right)\left(B + \sqrt{-\dfrac{\sigma_1\sigma_2}{\sigma_3}}\right), \end{cases}$$

and from (34), §24,

(24) $$\begin{cases} 2LL'M = BM^2 + B^3 + (N_1^2 + N_2^2 + N_3^2)B - 2N_1N_2N_3 \\ \quad = 2B^3 - B^2\dfrac{\wp''v}{i\wp'v} + \tfrac{1}{2}i\wp'v. \end{cases}$$

With

(25) $$N_\alpha = 0, \qquad aB^2 + L^2 + \sigma - s_\alpha = 0,$$

(26) $$Z_1 = \left(z - \frac{B}{M}\right)^2 - \left(\frac{N_\beta + N_\gamma}{M}\right)^2, \quad Z_2 = \left(z - \frac{B}{M}\right)^2 - \left(\frac{N_\beta - N_\gamma}{M}\right)^2,$$

(27) $$\wp 2w - e_\alpha = 0, \qquad \wp' 2w = 0,$$

(28) $$2b^3 - 3abc + a^2d = 0,$$

(29) $$\begin{cases} -aM^2 = L^2 + 2aL\sqrt{\dfrac{s_\beta - \sigma \cdot s_\gamma - \sigma}{s_\alpha - \sigma}} - s_\alpha + s_\beta + s_\gamma - \sigma \\ \quad = L^2 + 2aL\sqrt{\dfrac{\sigma_\beta\sigma_\gamma}{\sigma_\alpha}} - \sigma_\alpha + \sigma_\beta + \sigma_\gamma \\ \quad = \left(L + a\sqrt{\dfrac{\sigma_\beta\sigma_\gamma}{\sigma_\alpha}}\right)^2 - \dfrac{\sigma_\alpha - \sigma_\beta \cdot \sigma_\alpha - \sigma_\gamma}{\sigma_\alpha}, \end{cases}$$

27. With $\mu = 2n$, an even number, and a parameter

(1) $$v = \frac{\omega_3}{n}, \text{ or } \omega_1 + \frac{\omega_3}{n},$$

we can write

(2) $$\begin{cases} \psi' = \dfrac{1}{n}\cos^{-1}\dfrac{Hz^{n-1} + H_1z^{n-2} + \dots}{(-z^2+1)^{\frac{1}{2}n}}\sqrt{Z_1(-1)^n} \\ \quad = \dfrac{1}{n}\sin^{-1}\dfrac{Kz^{n-1} + K_1z^{n-2} + \dots}{(-z^2+1)^{\frac{1}{2}n}}\sqrt{aZ_2(-1)^n}, \end{cases}$$

and determine the coefficients from the differential relation of (38), §9.

It is of great assistance in the calculation to start with a knowledge of the the leading coefficients H and K, and these can be determined readily by taking $z = \infty$, when

(3) $$e^{n\psi' i} = H + K\sqrt{-a},$$

(4) $$e^{2n\psi' i} = \frac{H + K\sqrt{-a}}{H - K\sqrt{-a}},$$

since

(5) $$H^2 + K^2a = 1.$$

But now at $z=\infty$, from (18), §9,

$$(6) \qquad \psi i = \tfrac{1}{2} i I(2w, v) - \tfrac{1}{4}\log\frac{\wp(v_1-w)-\wp(v_2-w)}{\wp(v_1+w)-\wp(v_2+w)},$$

and from (25), (26), §9,

$$(7) \qquad \wp(v_1-w)-\wp(v_2-w) = -2\alpha\frac{B}{M} - 2\frac{L}{M}\sqrt{-a},$$

$$(8) \qquad \wp(v_1+w)-\wp(v_2+w) = -2\alpha\frac{B}{M} + 2\frac{L}{M}\sqrt{-a},$$

so that

$$(9) \qquad e^{2n\psi' i} = e^{nIi}\left(\frac{B\sqrt{-a}+L}{B\sqrt{-a}-L}\right)^{\frac{1}{2}n}$$

writing I for $I\,(2w, v)$.

With a parameter v as in (1), and

$$(10) \qquad M^2(\wp 2w - \wp v) = s - \sigma = aB^2 + L^2,$$

we have I given by an expression of the form (*Phil. Trans.*, 1904.)

$$(11) \qquad I = \frac{1}{n}\cos^{-1}\frac{G N_\beta N_\gamma}{(\sigma-s)^{\frac{1}{2}n}} = \frac{1}{n}\sin^{-1}\frac{F N_\alpha\sqrt{a}}{(\sigma-s)^{\frac{1}{2}n}},$$

$$(12) \qquad e^{nIi} = \frac{G N_\beta N_\gamma + F N_\alpha\sqrt{-a}}{(\sigma-s)^{\frac{1}{2}n}},$$

$$(13) \qquad e^{2nIi} = \frac{G N_\beta N_\gamma + F N_\alpha\sqrt{-a}}{G N_\beta N_\gamma - F N_\alpha\sqrt{-a}},$$

so that

$$(14) \qquad H + K\sqrt{-a} = \frac{\sqrt{(G N_\beta N_\gamma + F N_\alpha\sqrt{-a})}}{(B\sqrt{-a}+L)^{\frac{1}{2}n}},$$

$$(15) \qquad H - K\sqrt{-a} = \frac{\sqrt{(G N_\beta N_\gamma - F N_\alpha\sqrt{-a})}}{(B\sqrt{-a}-L)^{\frac{1}{2}n}},$$

whence H and K, in the form

$$(16) \qquad H = \frac{\cos}{\mathrm{ch}}\frac{1}{2}n(I+\beta), \quad K = \frac{\sin}{\mathrm{sh}}\frac{1}{2}n(I+\beta),$$

on putting

$$(17) \qquad B = \sqrt{s-\sigma}\,\frac{\cos}{\mathrm{ch}}\,\beta, \quad L = \sqrt{s-\sigma}\,\frac{\sin}{\mathrm{sh}}\,\beta.$$

When

(18) $$aB^2 + L^2 = 0, \ a = -1, \ B = L, \ s = \sigma,$$

(19) $$H + K = \frac{\sqrt{(2FN_\alpha)}}{(2L)^{\frac{1}{2}n}}, \ H - K = \frac{(2L)^{\frac{1}{2}n}}{\sqrt{(2FN_\alpha)}},$$

(20) $$H, K = \frac{\pm(2L)^n + 2FN_\alpha}{2(2L)^{\frac{1}{2}n}\sqrt{(2FN_\alpha)}}.$$

From the differential relation (37), §9, we find

(21) $$H_1 = H\frac{B + N_\alpha}{M} + nK\frac{L - P}{M}, \ K_1 = K\frac{B - N_\alpha}{M} - nH\frac{L - P}{M},$$

(22) $$\begin{cases} H_2 = H\left[\left(\frac{B + N_\alpha}{M}\right)^2 + \frac{1}{2}\left(\frac{N_\beta - N_\gamma}{M}\right)^2 - \frac{1}{2}n - \frac{1}{2}n^2\left(\frac{L - P}{M}\right)^2\right] \\ \quad + nK\left(-\frac{L'}{M} + \frac{2B + N_\alpha}{M}\frac{L - P}{M}\right), \\ K_2 = K\left[\left(\frac{B - N_\alpha}{M}\right)^2 + \frac{1}{2}\left(\frac{N_\beta + N_\gamma}{M}\right)^2 - \frac{1}{2}n - \frac{1}{2}n^2\left(\frac{L - P}{M}\right)^2\right] \\ \quad + nH\left(-\frac{L'}{M} + \frac{2B - N_\alpha}{M}\frac{L - P}{M}\right), \end{cases}$$

and so on.

28. Test by $v = \frac{1}{2}\omega_3$, $a = -1$, as in §18, from which the result in terms of z can be obtained by putting

$$t^2 = \frac{1 - z}{1 + z};$$

but independently, starting with z as the variable,

(1) $$\psi' = \frac{1}{2}\cos^{-1}\frac{Hz + H_1}{-z^2 + 1}\sqrt{Z_1} = \frac{1}{2}\sin^{-1}\frac{Kz + K_1}{-z^2 + 1}\sqrt{-Z_2},$$

(2) $$Z_1 = \left(z - \frac{B + N_3}{M}\right)^2 - \left(\frac{N_1 - N_2}{M}\right)^2$$

(3) $$Z_2 = \left(z - \frac{B - N_3}{M}\right)^2 - \left(\frac{N_1 + N_2}{M}\right)^2,$$

(4) $$-B^2 + L^2 = s - \sigma = s + 1,$$

so that

(5) $$N_3^2 = B^2 - L^2 + 1 = -s = b^2,$$

suppose

(6) $$N_1^2 = b^2 + \frac{1}{o^2},\ N_2^2 = b^2 + o^2,$$

and with

(7) $$\frac{1}{o} + o = 2P,$$

(8) $$e^{2R} = \frac{N_1 N_2 + 2Pb}{b^2 - 1} = \frac{\sqrt{(N_1 N_2 + b^2 - 1)} + \sqrt{(N_1 N_2 - b^2 + 1)}}{\sqrt{(N_1 N_2 + b^2 - 1)} - \sqrt{(N_1 N_2 - b^2 + 1)}},$$

(9) $$H + K = \frac{\sqrt{(N_1 N_2 + b^2 - 1)} + \sqrt{(N_1 N_2 - b^2 + 1)}}{\sqrt{2(B - L)}},$$

(10) $$H - K = \frac{\sqrt{(N_1 N_2 + b^2 - 1)} - \sqrt{(N_1 N_2 - b^2 + 1)}}{\sqrt{2(B + L)}},$$

whence H and K; and

(11) $$M^2 = 2L^2 - B^2 + 3b^2 + 4P^2 - 2 + 2\frac{\sqrt{[(b^2 - 1)^2 + 4P^2b^2]} + 2PL}{B^2 - L^2}.$$

Then

(12) $$Hz + H_1 = H\left(z + \frac{B + N_3}{M}\right) - 2K\frac{L - P}{M},$$

(13) $$Kz + K_1 = K\left(z + \frac{B - N_3}{M}\right) - 2H\frac{L - P}{M},$$

and the verification can be completed.

With

(14) $$B^2 - L^2 = 0,\ B = L,\ b = 1,$$

(15) $$N_1^2 = 1 + \frac{1}{o^2},\ N_2^2 = 1 + o^2,\ N_3^2 = 1,$$

(16) $$LM^2 = \left(L - \frac{1}{o}\right)(L - o)\left(L - \frac{1}{o} - o\right) = (L^2 - 2PL + 1)(L - 2P),$$

(17) $$H, K = \frac{\pm L^2 + P}{L\sqrt{2P}}.$$

Try

(18) $$H = 1,\ K = 0,\ P = L^2,$$

(19) $$M^2 = (L - 1)(2L - 1)(2L^2 + L + 1),$$

with

(20) $$N_3 = 0,\ b = 0,\ s = s_3,\ L^2 - B^2 = 1,\ N_1 = \frac{1}{o},\ N_2 = o,$$

(21) $$H = L,\ K = B,\ M^2 = (L - 2P)^2 - 1.$$

In Kirchhoff's case of

(22) $B=0,\quad L^2=1,\quad L=-1,\quad L'=0,\quad M^2=4P(P+1),\quad H=1,\quad K=0,$

(23) $\psi' = \frac{1}{2}\cos^{-1}\dfrac{z\sqrt{(z^2-1+\frac{1}{P})}}{-z^2+1} = \frac{1}{2}\sin^{-1}\dfrac{\sqrt{(-\frac{P+1}{P}z^2+1)}}{-z^2+1},$

satisfying

(24) $$\frac{d\psi'}{dz} = \frac{-(P+1)z^2+P-1}{M(-z^2+1)\sqrt{Z}}.$$

A change of sign of o^2 will change the parameter v from $\frac{1}{2}\omega_3$ to $\omega_1+\frac{1}{2}\omega_3$, still with $a=-1$; and now

(25) $$N_1^2 = B^2 - L^2 + \frac{1}{o^2} - 1,\quad N_2^2 = B^2 - L^2 - 1 + o^2,\quad N_3^2 = B^2 - L^2 - 1,$$

(26) $\dfrac{1}{o} - o = 2P.$

As a special case, take

(27) $$N_3 = 0,\qquad B^2 - L^2 = 1.$$

Again, still further, take

(28) $$L=0,\qquad B^2=1,\qquad H=1,\qquad K=0.$$

(29) $$\begin{cases} \psi' = \frac{1}{2}\cos^{-1}\dfrac{\left(z+\frac{B}{M}\right)\sqrt{\left[\left(z-\frac{B}{M}\right)^2 - 4\frac{P^2}{M^2}\right]}}{-z^2+1} \\[2ex] \quad = \frac{1}{2}\sin^{-1}\dfrac{\frac{2P}{M}\sqrt{\left[-\left(z-\frac{B}{M}\right)^2 + 4\frac{P^2+1}{M^2}\right]}}{-z^2+1}, \end{cases}$$

(30) $$M^2 = \frac{1}{o^2} - 1 + o^2 = 4P^2 + 1,\qquad \frac{L'}{M} = -\frac{2BP}{4P^2+1}.$$

This case can be written, putting $\dfrac{B}{M} = b$,

(31) $$\begin{aligned}(-z^2+1)e^{2\psi' i} &= (z+b)\quad \sqrt{[(z-b)^2 - 1 + b^2]} \\ &\quad + i\sqrt{(1-b^2)}\sqrt{[-(z-b)^2 + 1 + 3b^2]}.\end{aligned}$$

29. Next, with

(1) $$v = \omega_1 + \tfrac{1}{2}\omega_3,\quad a = 1,$$

(2) $$N_3^2 = B^2 + L^2 + 1 = b^2,$$

suppose; so that

(3) $$N_1^2 = B^2 + L^2 - \frac{1}{o^2} + 1 = b^2 - \frac{1}{o^2},$$

(4) $$N_2^2 = B^2 + L^2 + 1 - o^2 = b^2 - o^2.$$

Now we find, with

(5) $$\frac{1}{o}-o=2P,\ \sqrt{(b^2-1+N_1N_2)}=\sqrt{\frac{b^2+2Pb-1}{2}}+\sqrt{\frac{b^2-2Pb-1}{2}},$$

(6) $$H-Ki=\frac{(1+i)\sqrt{(b^2+2Pb-1)}+(1-i)\sqrt{(b^2-2Pb-1)}}{2(B-Li)},$$

(7) $$H+Ki=\frac{(1-i)\sqrt{(b^2+2Pb-1)}+(1+i)\sqrt{(b^2-2Pb-1)}}{2(B+Li)},$$

(8) $$H=\frac{(-B+L)\sqrt{(b^2+2Pb-1)}-(B+L)\sqrt{(b^2-2Pb-1)}}{2(b^2-1)},$$

(9) $$K=\frac{(B+L)\sqrt{(b^2+2Pb-1)}-(B-L)\sqrt{(b^2-2Pb-1)}}{2(b^2-1)},$$

(10) $$\psi'=\tfrac{1}{2}\cos^{-1}\frac{Hz+H_1}{-z^2+1}\sqrt{Z_1}=\tfrac{1}{2}\sin^{-1}\frac{Kz+K_1}{-z^2+1}\sqrt{Z_2},$$

(11) $$\begin{cases} Z_1=\left(z-\dfrac{B+N_3}{M}\right)^2-\left(\dfrac{N_1-N_2}{M}\right)^2, \\ Z_2=\left(z-\dfrac{B-N_3}{M}\right)^2-\left(\dfrac{N_1+N_2}{M}\right)^2, \end{cases}$$

(12) $$\begin{cases} M^2=-2L^2-B^2+N_1^2+N_2^2+N_3^2+2\dfrac{BN_1N_2N_3-2PL}{B^2+L^2} \\ \quad =-L^2+2b^2-4P^2-1+2\dfrac{Bb\sqrt{[(b^2-1)^2-4P^2b^2]}-2PL}{b^2-1}. \end{cases}$$

From the differential relation (38), §9

(13) $$Hz+H_1=H\left(z+\frac{B+N_3}{M}\right)-2K\frac{L-P}{M},$$

(14) $$Kz+K_1=K\left(z+\frac{B-N_3}{M}\right)+2H\frac{L-P}{M}.$$

As a special case, try

(15) $$N_1=0,\quad B^2+L^2=\frac{1}{o^2}-1,\quad N_3=b=\frac{1}{o},$$

(16) $$H=\frac{(-B+L)o}{\sqrt{2}\sqrt{(1-o^2)}},\quad K=\frac{(B+L)o}{\sqrt{2}\sqrt{(1-o^2)}},$$

(17) $$M^2=\frac{1}{o^2}+1-(L+o)^2;$$

and then in Kirchhoff's case

$$(18)\qquad B=0,\quad H=K=\frac{1}{\sqrt{2}},\quad L=\sqrt{\left(\frac{1}{o^2}-1\right)},\quad M=1-\sqrt{1-o^2}.$$

$$(19)\begin{cases}\psi=\frac{1}{2}\cos^{-1}\dfrac{\left(z+\dfrac{1-\sqrt{1-o^2}}{o}\right)\sqrt{\left[z^2-2z\dfrac{1+\sqrt{1-o^2}}{o^3}+\left(\dfrac{1+\sqrt{1-o^2}}{o}\right)^2\right]}}{\sqrt{2}(-z^2+1)}\\[2ex] \quad=\frac{1}{2}\sin^{-1}\dfrac{\left(z-\dfrac{1-\sqrt{1-o^2}}{o}\right)\sqrt{\left[z^2+2z\dfrac{1+\sqrt{1-o^2}}{o^3}+\left(\dfrac{1+\sqrt{1-o^2}}{o}\right)^2\right]}}{\sqrt{2}(-z^2+1)}\end{cases}$$

leading on differentiation to

$$(20)\qquad \frac{d\psi'}{dz}=\frac{\dfrac{\sqrt{1-o^2}}{2o^3}\left[(1+o^2+\sqrt{1-o^2})z^2+3-o^2+3\sqrt{1-o^2}\right]}{(-z^2+1)\sqrt{Z}},$$

so that $L'=0$.

But if $L=0$ instead of B when $N_1=0$,

$$(21)\qquad -H=K=\frac{1}{\sqrt{2}},\quad B=\sqrt{\left(\frac{1}{o^2}-1\right)},\quad M^2=\frac{1}{o^2}+1-o^2,$$

$$(22)\begin{cases}\psi'=\frac{1}{2}\cos^{-1}\dfrac{\left(-z-\dfrac{o^2+\sqrt{1-o^2}}{\sqrt{1+o^2-o^4}}\right)\sqrt{\left[z^2-2z\dfrac{1+\sqrt{1-o^2}}{\sqrt{1+o^2-o^4}}+\dfrac{1-o^2+o^4+2\sqrt{1-o^2}}{1+o^2-o^4}\right]}}{\sqrt{2}(-z^2+1)}\\[2ex] \quad=\frac{1}{2}\sin^{-1}\dfrac{\left(z-\dfrac{o^2-\sqrt{1-o^2}}{\sqrt{1+o^2-o^4}}\right)\sqrt{\left[z^2+2z\dfrac{1-\sqrt{1-o^2}}{\sqrt{1+o^2-o^4}}+\dfrac{1-o^2+o^4-2\sqrt{1-o^2}}{1+o^2-o^4}\right]}}{\sqrt{2}(-z^2+1)},\end{cases}$$

and

$$(23)\qquad \frac{d\psi'}{dz}=\frac{-z^2\dfrac{1-o^2}{2\sqrt{1+o^2-o^4}}-2z\dfrac{o^2\sqrt{1-o^2}}{1+o^2-o^4}+\dfrac{1-o^2}{2\sqrt{1+o^2-o^4}}}{(-z^2+1)\sqrt{Z_1Z_2}}.$$

Replacing o^2 by k,

$$(24)\begin{cases}\psi'=\frac{1}{2}\cos^{-1}\dfrac{\left(-z-\dfrac{k+\sqrt{1-k}}{\sqrt{1+k-k^2}}\right)\sqrt{\left[z^2-2z\dfrac{1+\sqrt{1-k}}{\sqrt{1+k-k^2}}+\dfrac{1-k+k^2+2\sqrt{1-k}}{1+k-k^2}\right]}}{\sqrt{2}(-z^2+1)}\\[2ex] \quad=\frac{1}{2}\sin^{-1}\dfrac{\left(z-\dfrac{k-\sqrt{1-k}}{\sqrt{1+k-k^2}}\right)\sqrt{\left[z^2+2z\dfrac{1-\sqrt{1-k}}{\sqrt{1+k-k^2}}+\dfrac{1-k+k^2-2\sqrt{1-k}}{1+k-k^2}\right]}}{\sqrt{2}(-z^2+1)},\end{cases}$$

$$(25)\qquad \frac{d\psi'}{dz}=\frac{-z^2\dfrac{1-k}{2\sqrt{1+k-k^2}}-2z\dfrac{k\sqrt{1-k}}{1+k-k^2}+\dfrac{1-k}{2\sqrt{1+k-k^2}}}{(-z^2+1)\sqrt{Z}},$$

$$(26)\qquad L'=\sqrt{\frac{1-k}{1+k-k^2}},\quad L=0.$$

30. With $a=1$ and a parameter

(1) $$v=\omega_1+\tfrac{1}{3}\omega_3,$$

(2) $$N_a^2=B^2+L^2+\sigma-s_a=s-s_a, \text{ with } B^2+L^2=s-\sigma,$$

and, as in §22,

(3) $$\sigma=2c,\quad \sigma-s_2=2c-1,\quad \sqrt{(s_1-\sigma.\sigma-s_3)}=c\sqrt{(2c-1)},$$
$$\sqrt{-\Sigma}=2c(2c-1),\quad P=\tfrac{1}{3}(2c-1),$$

so that

(4) $$N_2^2=s-s_2=s-1,$$

(5) $$N_1^2N_3^2=(s-s_1)(s-s_3)=s^2-(c^2+2c)s+c^2.$$

As in §21 we can put

(6) $$\frac{s-s_2}{s-\sigma}=q^2,\quad s-\sigma=\frac{2c-1}{q^2-1},\quad s=\frac{2cq^2-1}{q^2-1},$$

and then

(7) $$N_2^2=(2c-1)\frac{q^2}{q^2-1},\quad N_1^2N_3^2=(2c-1)\frac{-Q_1Q_2}{(q^2-1)^2},$$

with Q_1, Q_2 as in (16), §21; and now for $\mu=6$, as in §22,

(8) $$e^{ai}=\left[\sqrt{\frac{q+1}{2}}\sqrt{Q_1}+i\sqrt{\frac{q-1}{2}}\sqrt{-Q_2}\right]e^{\frac{3}{2}\beta i},$$

(9) $$Q_1,\ Q_2=-cq\pm cq+1.$$

Now we have

(10) $$\begin{cases}\psi'=\tfrac{1}{3}\cos^{-1}\dfrac{Hz^2+H_1z+H_2}{(-z^2+1)^{\frac{3}{2}}}\sqrt{(-Z_1)}\\[2ex] \quad=\tfrac{1}{3}\sin^{-1}\dfrac{Kz^2+K_1z+K_2}{(-z^2+1)^{\frac{3}{2}}}\sqrt{(-Z_2)},\end{cases}$$

(11) $$Z_1=\left(z-\frac{B+N_2}{M}\right)^2-\left(\frac{N_1-N_3}{M}\right)^2,\ Z_2=\left(z-\frac{B-N_2}{M}\right)^2-\left(\frac{N_1+N_3}{M}\right)^2$$

(12) $$M^2=-L^2+2Lc(q^2-1)-c^2+1-2B\frac{q\sqrt{(-Q_1Q_2)}}{\sqrt{(q^2-1)}}.$$

But with $a=-1$,

(13) $$\psi'=\tfrac{1}{3}\cos^{-1}\frac{Hz^2+H_1z+H_2}{(-z^2+1)^{\frac{3}{2}}}\sqrt{-Z_1}$$
$$=\tfrac{1}{3}\sin^{-1}\frac{Kz^2+K_1z+K_2}{(-z^2+1)^{\frac{3}{2}}}\sqrt{Z_2},$$

with

(14) $$H^2 - K^2 = 1, \quad H = \text{ch}\alpha, \quad K = \text{sh}\alpha,$$

(15) $$-B^2 + L^2 = s - \sigma, \quad B^2 - L^2 = \sigma - s,$$

(16) $$B = \sqrt{s-\sigma}\ \text{sh}\beta, \text{ or } \sqrt{\sigma - s}\ \text{ch}\beta,$$

(17) $$L = \sqrt{s-\sigma}\ \text{ch}\beta, \text{ or } \sqrt{\sigma - s}\ \text{sh}\beta,$$

and

(18) $$e^{\pm\alpha} = \left(\sqrt{\frac{q+1}{2}}\sqrt{Q_1} \pm \sqrt{\frac{q-1}{2}}\sqrt{Q_2}\right) e^{\pm\frac{3}{2}\beta}.$$

The determination of H_1, K_1, H_2, K_2 can then be carried out by (22), (23), §24, and the verification effected, but the presence of M makes the expressions complicated, much more so than for the previous form of solution in §19, so we do not proceed further with this or higher values of n, with z as variable.

31. A similar procedure will determine the leading coefficient H and K when μ is odd, and ψ' is expressed in the form of equation (2), §12, although there is no need to carry out the rest of the calculation now that the degree can be halved by changing to the stereographic projection with the new variable $t = \tan \frac{1}{2}\theta$ in the place of $z = \cos\theta$.

In this case $a = -1$, and taking the parameter

(1) $$v = \frac{4\omega_3}{\mu}, \text{ so that } s(v) = 0,$$

and putting

(2) $$-B^2 + L^2 = s(2w) - s(v) = c^2,$$

we can write

(3) $$I = I(2w, v) = \frac{2}{\mu}\,\text{ch}^{-1}\frac{F_1\sqrt{C_1}}{c^{\frac{1}{2}\mu}} = \frac{2}{\mu}\,\text{sh}^{-1}\frac{F_2\sqrt{C_2}}{c^{\frac{1}{2}\mu}},$$

(4) $$e^{\frac{1}{2}\mu I i} = \frac{F_1\sqrt{C_1} + F_2\sqrt{C_2}}{c^{\frac{1}{2}\mu}},$$

(5) $$e^{\mu I i} = \frac{F_1\sqrt{C_1} + F_2\sqrt{C_2}}{F_1\sqrt{C_1} - F_2\sqrt{C_2}},$$

(6) $$H + K = \frac{F_1\sqrt{C_1} + F_2\sqrt{C_2}}{2(L+B)^{\frac{1}{2}\mu}},$$

(7) $$-H + K = \frac{F_1\sqrt{C_1} - F_2\sqrt{C_2}}{2(L-B)^{\frac{1}{2}\mu}}$$

But if $L = B$, $c = 0$, and

$$(8) \qquad H,\ K = \frac{F_1^2 xy \mp (2L)^\mu}{(2L)^{\frac{1}{2}\mu} F_1 \sqrt{xy}},$$

$$(9) \qquad \begin{cases} M^2 = L^2 + N_1^2 + N_2^2 + N_3^2 + \dfrac{B^2 \wp'' v - \frac{1}{2}\wp'^2 v}{Bi\wp' v} \\ \quad = L^2 + s_1 + s_2 + s_3 + \text{—— : : : : ——}. \end{cases}$$

Take for example

$$(10) \qquad \begin{cases} \mu = 3\,,\ v = \dfrac{4\omega}{3}\,, \\ S = 4s^3 - (s - m)^2, \\ s_1 + s_2 + s_3 = \frac{1}{4}\,, \\ s_2 s_3 + s_3 s_1 + s_1 s_2 = \frac{1}{2} m = \frac{1}{2} \wp'' v\,, \\ s_1 s_2 s_3 = \frac{1}{4} m^2 = -\frac{1}{4} \wp'^2 v,\ i\wp' v = m, \\ H,\ K = \dfrac{m \mp (2L)^3}{(2L)^{\frac{3}{2}} \sqrt{m}}, \\ M^2 = (L + \frac{1}{2})^2 + \dfrac{m}{2L}\,, \\ Z = -\left(z - \dfrac{L}{M}\right)^4 + \dfrac{1}{2M^2}\left(z - \dfrac{L}{M}\right)^2 \\ \qquad\qquad - 4\dfrac{m}{M^3}\left(z - \dfrac{L}{M}\right) - \dfrac{1 - 32m}{16M^4}\,; \end{cases}$$

or, since

$$(11) \qquad m = 2LM^2 - 2L(L + \tfrac{1}{2})^2$$

$$(12) \qquad \begin{cases} Z = -\left(z - \dfrac{L}{M}\right)^4 + \dfrac{1}{2M^2}\left(z - \dfrac{L}{M}\right)^2 \\ \qquad - \dfrac{\delta LM^2 - \delta L(L + \frac{1}{2})^2}{M^2}\left(z - \dfrac{L}{M}\right) \\ \qquad - \dfrac{1}{16M^4} + \dfrac{4LM^2 - 4L(L + \frac{1}{2})^2}{M^4}. \end{cases}$$

Putting $z = \pm 1$,

$$(13) \qquad \begin{cases} Z = -4\left(\dfrac{L' \mp L}{M}\right)^2 \\ \quad = -\left(\dfrac{3L^2 + 2L + \frac{1}{4}}{M^2} \mp 2\dfrac{L}{M} - 1\right)^2, \end{cases}$$

so that

(14) $$2\frac{L'}{M}=\frac{3L^2+2L+\frac{1}{4}}{M^2}-1$$

(15) $$L'+L=\frac{(3L+\frac{1}{2}-M)(M+L+\frac{1}{2})}{2M},$$

$$L-L'=\frac{(3L+\frac{1}{2}+M)(M-L-\frac{1}{2})}{2M},$$

(16) $$Z+4\left(\frac{Lz-L'}{M}\right)^2=-(z^2-1)\left(z^2-4\frac{L}{M}z-1+E\right)$$

(17) $$E=\frac{4L^2-1}{2M^2}+2.$$

In the result of §14 we have now to take

(18) $$\gamma=-\tfrac{1}{2},\ k=\frac{3L+\frac{1}{2}+M}{3L+\frac{1}{2}-M}\sqrt{\frac{M-L-\frac{1}{2}}{M+L+\frac{1}{2}}}.$$

Try

(19) $$m=-(2L)^3,\ K=0,$$

(20) $$M^2=\tfrac{1}{4}(1+6L)(1-2L)$$

(21) $$4M-1-4L=-(\sqrt{1+6L}-\sqrt{1-2L})^2$$

(22) $$4M+1+4L=\ (\sqrt{1+6L}+\sqrt{1-2L})^2,$$

so that k is imaginary.

Try

(23) $$m=(2L)^3,\ H=0,$$

(24) $$M^2=(L+\tfrac{1}{2})^2+4L^2$$

(25) $$4M^2-(2L+1)^2=16L^2$$

(26) $$\lambda^2=k=\left(\frac{\sqrt{2M+2L+1}+\sqrt{2M-2L-1}}{\sqrt{2L+1}}\right)^2.$$

Thus, as a numerical case, with

(27) $$L=1,\ M=\frac{5}{2},\ m=8,\ L'=-\frac{1}{5},\ k=3,$$

(28) $$Z=-z^4+\frac{8}{5}z^3-\frac{22}{25}z^2-\frac{232}{125}z+\frac{759}{625},$$

$Z=0$ having two real roots between ± 1; and we find

$$(29) \qquad \psi' = \tfrac{1}{3}\cos^{-1}\frac{\frac{7}{5}z^2+\frac{4}{5}z-\frac{59}{125}}{(\sqrt{-z^2+1})^{\frac{3}{2}}} = \tfrac{1}{3}\sin^{-1}\frac{\left(z+\frac{4}{5}\right)\sqrt{Z}}{(-z^2+1)^{\frac{3}{2}}},$$

or changing to the stereographic projection,

$$(30) \qquad 10\sqrt{5}(te^{\psi' i})^{\frac{3}{2}} = (t-3)\sqrt{(\ \ 4t^4+24t^3-5t^2+4t+6)} + i(t+3)\sqrt{(-4t^4+24t^3+5t^2+4t-6)}.$$

For the motion of the centre

$$(31) \qquad \rho^2 = \left(z-\frac{4}{5}\right)^2 + \frac{3}{5},$$

and

$$(32) \qquad \rho^3 e^{3\omega' i} = z^3 - \frac{12}{5}z^2 + H_2 z + H_3 + iK_1\sqrt{Z},$$

$$(33) \qquad K_1 = 1,\ H_2 = \frac{83}{25},\ H_3 = -\frac{104}{125}.$$

The degree is halved by putting

$$(34) \qquad z - \frac{4}{5} = \sqrt{\frac{3}{5}\,\frac{2w}{w^2-1}},\ \rho = \sqrt{\frac{3}{5}\,\frac{w^2+1}{w^2-1}},$$

and then

$$(35) \qquad (\rho e^{\omega' i})^{\frac{3}{2}} = \frac{(w+1)\sqrt{W_1} + i(w-1)\sqrt{W_2}}{\sqrt[4]{(500)}\,(w^2-1)^{\frac{3}{2}}},$$

$$(36) \qquad W_1 = (4\sqrt{5}+3\sqrt{3})w^4 - 8(\sqrt{5}-\sqrt{3})w^3 - 10\sqrt{3}w^2 + 8(\sqrt{5}+\sqrt{3})w - 4\sqrt{5}+3\sqrt{3},$$

$$(37) \qquad W_2 = -(4\sqrt{5}-3\sqrt{3})w^4 - 8(\sqrt{5}+\sqrt{3})w^3 - 10\sqrt{3}w^2 + 8(\sqrt{5}-\sqrt{3})w + 4\sqrt{5}+3\sqrt{3}.$$

A numerical application, such as inserted here and elsewhere, is useful in showing the existence of a real solution, and the region of its reality, as well as in settling a doubtful sign in the general algebraical case.